

# INTEGRAL BASIS THEOREM OF CYCLOTOMIC KHOVANOV-LAUDA-ROUQUIER ALGEBRAS OF TYPE A

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**ABSTRACT.** In this paper we prove that the cyclotomic Khovanov-Lauda-Rouquier algebras in type A,  $\mathcal{R}_n^\Lambda$ , are  $\mathbb{Z}$ -free. We then extend the graded cellular basis of  $\mathcal{R}_n^\Lambda$  constructed by Hu and Mathas to  $\mathcal{R}_n$  and use this basis to give a classification of all irreducible  $\mathcal{R}_n$ -modules.

## 1. Introduction

Khovanov and Lauda [13, 12] and Rouquier [23] have introduced a remarkable new family of algebras  $\mathcal{R}_n$ , the **quiver Hecke algebras**, for each oriented quiver. They showed that these algebras categorify the positive part of the enveloping algebras of the corresponding quantum groups. The algebras  $\mathcal{R}_n$  are naturally  $\mathbb{Z}$ -graded. Varagnolo and Vasserot [24] proved that, under this categorification, the canonical basis of the positive part of the quantum group corresponds to the image of the projective indecomposable modules in the Grothendieck rings of the quiver Hecke algebras when the Cartan matrix is symmetric.

The algebra  $\mathcal{R}_n$  is infinite dimensional and for every highest weight vector in the corresponding Kac-Moody algebra there is an associated finite dimensional 'cyclotomic quotient'  $\mathcal{R}_n^\Lambda$  of  $\mathcal{R}_n$ . The cyclotomic quiver algebras  $\mathcal{R}_n^\Lambda$  were originally defined by Khovanov and Lauda [13, 12] and Rouquier [23] who conjectured that these algebras should categorify the irreducible representations of the corresponding quantum group. Lauda and Vazirani [18] proved that, up to shift, the simple  $\mathcal{R}_n$ -modules are indexed by the vertices of the corresponding crystal graph, and Kang and Kashiwara [11] proved the full conjecture by showing that the images of the projective irreducible modules in the Grothendieck ring  $\text{Rep}(\mathcal{R}_n^\Lambda)$  correspond to the canonical basis of the corresponding highest weight module. Prior to this work, Brundan and Stropple [6] proved this conjecture in the special case when  $\Lambda$  is a dominant weight of level 2 and  $\Gamma$  is the linear quiver and Brundan and Kleshchev [4] established the conjecture for all  $\Lambda$  when  $\Gamma$  is a quiver of type A.

Let  $\Gamma$  be the quiver of type  $A_e$ , for  $e \in \{0, 2, 3, 4, \dots\}$ . Brundan and Kleshchev [3] proved that every degenerate and non-degenerate cyclotomic Hecke algebra  $H_n^\Lambda$  of type  $G(r, 1, n)$  over a field is isomorphic to a cyclotomic quiver Hecke algebra  $\mathcal{R}_n^\Lambda$  of type A. They did this by constructing an explicit isomorphism between these two algebras.

The algebras  $\mathcal{R}_n^\Lambda$  are defined by generators and relations and so these algebras are defined over any integral domain. Hu and Mathas [9] defined a homogeneous basis  $\{\psi_{\text{st}}\}$  of the cyclotomic quiver algebras  $\mathcal{R}_n^\Lambda$  (see Theorem 2.37 below), and they showed that  $\mathcal{R}_n^\Lambda$  is  $\mathbb{Z}$ -free whenever  $e = 0$  or  $e$  is invertible in the ground ring. They asked whether the algebra  $\mathcal{R}_n^\Lambda$  is always  $\mathbb{Z}$ -free. Kleshchev-Mathas-Ram [14] defined  $\mathbb{Z}$ -free Specht modules for the cyclotomic KLR algebras of type A (and the affine KLR algebras of type A), but that the existence of these modules does not imply that the cyclotomic KLR algebras are torsion free. The main result of this paper shows that this is always the case. More precisely, we prove the following.

**Theorem 1.1.** *Let  $\mathcal{R}_n^\Lambda(\mathbb{Z})$  be a cyclotomic Khovanov-Lauda-Rouquier algebra of type A over  $\mathbb{Z}$ , where  $\Lambda$  is a dominant weight of height  $\ell$ . Then  $\mathcal{R}_n^\Lambda(\mathbb{Z})$  is a graded cellular algebra, with respect to the dominance order, with homogeneous cellular basis  $\{\psi_{\text{st}} \mid \lambda \in \mathcal{P}_n^\Lambda \text{ and } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)\}$ . In particular,  $\mathcal{R}_n^\Lambda(\mathbb{Z})$  is  $\mathbb{Z}$ -free of rank  $\ell^n n!$ .*

If  $\mathcal{O}$  is any integral domain then  $\mathcal{R}_n^\Lambda(\mathcal{O}) \cong \mathcal{R}_n^\Lambda(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}$ , so it follows that  $\mathcal{R}_n^\Lambda(\mathcal{O})$  is free over  $\mathcal{O}$ .

The proof of our main theorem is long and technical, requiring a delicate multistage induction. Fortunately, by [9, Theorem 5.14] we may assume that  $e \neq 2$ . Even though our arguments should apply in this case, being able to assume that  $e \neq 2$  dramatically simplifies our arguments because the quiver of type  $A_e$  is simply laced when  $e \neq 2$ .

The starting point for our arguments is the observation that the definition of Hu and Mathas' the homogeneous elements  $\psi_{\text{st}}$  makes sense over any ring. Consequently, the linearly independent elements  $\{\psi_{\text{st}}\}$  span a  $\mathbb{Z}$ -free submodule  $R_n^\Lambda$  of  $\mathcal{R}_n^\Lambda$ . To prove our Main Theorem it is therefore enough to show that  $R_n^\Lambda$  is closed multiplication by the generators of  $\mathcal{R}_n^\Lambda$  and that the identity element of  $\mathcal{R}_n^\Lambda$  belongs to  $R_n^\Lambda$ .

*Key words and phrases.* Cyclotomic Hecke algebras, Khovanov-Lauda-Rouquier algebras, Cellular basis.

The algebra  $\mathcal{R}_n^\Lambda$  is generated by elements  $y_r$ ,  $\psi_s$  and  $e(\mathbf{i})$ , where  $1 \leq r \leq n$ ,  $1 \leq s < n$  and  $\mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n$ . To prove our Main Theorem we need to treat these three classes of generators separately. The cellular basis element  $\psi_{\text{st}}$  is indexed by two standard  $\lambda$ -tableaux where  $\lambda$  is a multipartition of  $n$ ; the definitions of these terms are recalled in Section 1. We argue by simultaneous induction on  $n$ , and on the lexicographic orderings on the set of multipartitions, to show that multiplication by the KLR generators always sends  $\psi_{\text{st}}$  to a  $\mathbb{Z}$ -linear combination of terms  $\psi_{\text{uv}}$  which are larger in the lexicographic order. Multiplication by  $y_r$  is the hardest case, partly because once this case is understood it can be used to describe the action of  $\psi_r$  and  $e(\mathbf{i})$  on the  $\psi$ -basis of  $\mathcal{R}_n^\Lambda$ .

As a consequence of Theorem 1.1, we obtain a graded cellular basis of  $\mathcal{R}_n^\Lambda(\mathbb{Z})$ . We then extend this basis to obtain a graded cellular basis of  $\mathcal{R}_n$ , and hence  $\mathcal{R}_n$  is a graded cellular algebra. Hence we can use similar arguments to Graham-Lehrer [7] to give a complete set of non-isomorphic graded irreducible  $\mathcal{R}_n$ -modules. Koenig and Xi [17] introduced the notion of affine cellular algebras and they have shown that the affine Hecke algebra of type A is an affine cellular algebra. The work of Koenig and Xi predates the KLR grading, but nonetheless it gives a classification of the irreducible representation of affine Hecke algebras. Our approach is very different to that of Koenig and Xi in that, first, we incorporate the grading and, secondly, we obtain a new labeling of the irreducible representations that is compatible the labeling of the irreducible modules of the cyclotomic quotients.

In more detail, this paper is organized as follows. In Section 1 we summarise the background material from the representation theory of the cyclotomic Khovanov-Lauda-Rouquier algebras that we need, including the theory of (graded) cellular algebras and the combinatorics of multipartitions and tableaux. In Section 2 considers the special case where  $\lambda$  is a multicomposition that has at most two rows. Once this case is understood we show for an arbitrary multipartition  $\lambda$  that  $\psi_{\text{st}} y_r$  is a  $\mathbb{Z}$ -linear combination of higher terms, where  $\text{st}$  is the ‘initial’  $\lambda$ -tableau. Section 3 begins by proving, again by induction, that  $\psi_{\text{st}} y_r$  is a linear combination of bigger terms in  $\mathcal{R}_n^\Lambda$ . By considering the Garnir tableau of two-rowed multipartition we then show that  $\psi_{\text{st}} y_r$  can be written in the required form. This result is then extended to multipartitions of arbitrary shape. Finally, we deduce that  $e(\mathbf{i}) \in \mathcal{R}_n^\Lambda$ , for all  $\mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n$ , which completes the proof of our main result. In Section 4 we define a sequence of weights  $\Lambda^\infty$  and using it to extend the graded cellular basis of  $\mathcal{R}_n^\Lambda$  to  $\mathcal{R}_n$  and hence define a graded cellular basis for  $\mathcal{R}_n$ . By adapting the arguments of Graham-Lehrer [7] we give a complete set of non-isomorphic graded simple  $\mathcal{R}_n$ -modules.

Finally, we remark that the calculations in Sections 2 and 3 gives an algorithm inductively for computing  $\psi_{\text{st}} y_r$  and  $\psi_{\text{st}} \psi_r$ . We extend the techniques developed in this paper to give a KLR grading of the Brauer algebras [19].

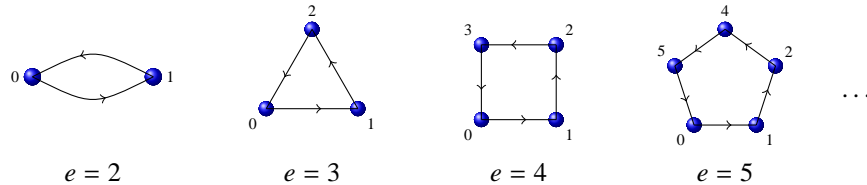
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## 2. Khovanov-Lauda-Rouquier Algebras

In this section we are going to introduce the necessary background for our work. First we will define our principal object of study — the (cyclotomic) Khovanov-Lauda-Rouquier algebras  $\mathcal{R}_n^\Lambda$ . Then we give a brief introduction to (graded) cellular algebras and symmetric groups. Finally after explaining tableaux combinatorics we describe a graded cellular basis for the cyclotomic KLR algebra, found by Hu and Mathas [9].

### 2.1. The cyclotomic Khovanov-Lauda-Rouquier algebras

Fix an integer  $e \in \{0, 2, 3, 4, \dots\}$  and  $I = \mathbb{Z}/e\mathbb{Z}$ . Let  $\Gamma_e$  be the oriented quiver with vertex set  $I$  and directed edges  $i \rightarrow i+1$ , for  $i \in I$ . Thus,  $\Gamma_e$  is the quiver of type  $A_\infty$  if  $e = 0$  and if  $e \geq 2$  then it is a cyclic quiver of type  $A_e^{(1)}$ :



Let  $(a_{i,j})_{i,j \in I}$  be the symmetric Cartan matrix associated with  $\Gamma_e$ , so that

$$a_{i,j} = \begin{cases} 2, & \text{if } i = j, \\ 0, & \text{if } i \neq j \pm 1, \\ -1, & \text{if } e \neq 2 \text{ and } i = j \pm 1, \\ -2, & \text{if } e = 2 \text{ and } i = j + 1. \end{cases}$$

for  $\mathbf{i} \in I^n$ ,  $1 \leq r < n$  and  $1 \leq s \leq n$ . The  $r$ -th string of the diagram is the string labelled with  $i_r$ .

Diagrams are considered up to isotopy, and multiplication of diagrams is given by concatenation, subject to the relations (2.2)–(2.9). In more detail, if  $D_1$  and  $D_2$  are two diagrams then the diagrammatic analogue of the relation  $e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}}e(\mathbf{i})$  is

$$D_1 \cdot D_2 = \begin{array}{c} \boxed{D_1} \\ \vdots \\ \boxed{D_2} \end{array} = \delta_{\mathbf{ij}} \begin{array}{c} \boxed{D_1} \\ \boxed{D_2} \end{array}$$

(The first diagram shows  $D_1$  and  $D_2$  connected by vertical lines labeled  $i_1, i_2, \dots, i_n$  on the left and  $j_1, j_2, \dots, j_n$  on the right. The second diagram shows  $D_1$  and  $D_2$  concatenated vertically, with labels  $i_1, i_2, \dots, i_n$  on the left of  $D_1$  and  $j_1, j_2, \dots, j_n$  on the left of  $D_2$ .)

That is,  $D_1 \cdot D_2 = 0$  unless the labels of the strings on the bottom of  $D_1$  match the corresponding labels on the top of the strings in  $D_2$  in which case we just concatenate the two diagrams.

Multiplication by  $y_r$  simply adds a decorative dot to the  $r$ -th string, reading left to right, so relations (2.3)–(2.5) become self when written in terms of diagrams. Ignoring the extraneous strings on the left and right, and setting  $i = i_r$  and  $j = i_{r+1}$ , the diagrammatic analogue of relations (2.6) and (2.7) is

$$\begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} - \begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \delta_{ij} \begin{array}{c} i \\ | \\ j \end{array} = \begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} - \begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}. \quad (2.10)$$

Similarly, if  $e \neq 2$  then relation (2.8) becomes

$$\begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \begin{cases} 0, & \text{if } i = j, \\ \begin{array}{c} i \quad j \\ | \quad | \end{array}, & \text{if } i \neq j \pm 1, \\ \pm \begin{array}{c} i \quad j \\ | \quad | \end{array} \mp \begin{array}{c} i \quad j \\ | \quad | \end{array}, & \text{if } j = i \pm 1. \end{cases} \quad (2.11)$$

(The diagrams for  $j = i \pm 1$  show a crossing with a dot on the  $i$ -th string.)

and if  $e \neq 2$  then the diagrammatic analogue of relation (2.9) is

$$\begin{array}{c} i \quad j \quad k \\ \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \end{array} - \begin{array}{c} i \quad j \quad k \\ \diagup \quad \diagdown \quad \diagup \\ \diagdown \quad \diagup \quad \diagdown \end{array} = \delta_{i,k}(\delta_{i,j+1} - \delta_{i,j-1}) \begin{array}{c} i \quad j \quad k \\ | \quad | \quad | \end{array}. \quad (2.12)$$

Using the relations in  $\mathcal{R}_n(\mathcal{O})$  it is easy to verify the following identity which we record for future use:

$$\hat{e}(\mathbf{i})\hat{y}_r^k\hat{y}_{r+1}^k\hat{\psi}_r = \hat{e}(\mathbf{i})\hat{\psi}_r\hat{y}_r^k\hat{y}_{r+1}^k \quad (2.13)$$

for any  $\mathbf{i}$ . Clearly it is enough to prove this relation when  $k = 1$  when, diagrammatically, this identity takes the form

$$\begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad (2.14)$$

(The diagrams show a crossing with dots on the  $i$ -th and  $j$ -th strings.)

locally on the  $r$  and  $r + 1$ -th strings and where we set  $i = i_r$  and  $j = i_{r+1}$ .

Three more easy, and very useful, consequences of the relations are the following:

$$\begin{array}{c}
 \begin{array}{c} i \\ | \end{array} = - \begin{array}{c} i \quad i \\ \diagup \quad \diagdown \\ | \quad | \end{array} - \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ | \quad | \end{array} - \begin{array}{c} i \quad i \\ \diagup \quad \diagdown \\ | \quad | \end{array} \quad (2.15)
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} i \\ | \end{array} = - \begin{array}{c} i \quad i \\ \diagup \quad \diagdown \\ | \quad | \end{array} - \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ | \quad | \end{array} - \begin{array}{c} i \quad i \\ \diagup \quad \diagdown \\ | \quad | \end{array} - \begin{array}{c} i \\ | \end{array} \quad (2.16)
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ | \quad | \end{array} \stackrel{(2.10)}{=} \begin{array}{c} i \quad i \\ \diagup \quad \diagdown \\ | \quad | \end{array} - \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ | \quad | \end{array} \stackrel{(2.11)}{=} \begin{array}{c} i \quad i \\ \diagup \quad \diagdown \\ | \quad | \end{array} - \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ | \quad | \end{array} \quad (2.17)
 \end{array}$$

Note that (2.16) follows by multiplying (2.15) by  $y_{r+1}$  and expanding.

In the rest of the paper we will play around with these diagrammatic notations a lot. In order to make the reader easy to follow our calculation we will use dotted strands to represent moving strands and arrows to represent moving dots. If we are going to move a dot then we will also write the strand which the dot is on dotted so the reader can see the arrow clearly. For example, we will write

$$\begin{array}{c}
 \begin{array}{c} 1 \quad 2 \quad 1 \quad 3 \quad 0 \quad 1 \\ \diagdown \quad \diagup \\ | \quad | \end{array} \stackrel{(2.12)}{=} \begin{array}{c} 1 \quad 2 \quad 1 \quad 3 \quad 0 \quad 1 \\ \diagup \quad \diagdown \\ | \quad | \end{array} - \begin{array}{c} 1 \quad 2 \quad 1 \quad 3 \quad 0 \quad 1 \\ | \quad | \end{array}
 \end{array}$$

to signify the application of relation (2.12) and

$$\begin{array}{c}
 \begin{array}{c} 1 \quad 2 \quad 1 \quad 3 \quad 3 \quad 0 \\ | \quad | \quad | \quad | \quad | \end{array} \stackrel{(2.14)}{=} \begin{array}{c} 1 \quad 2 \quad 1 \quad 3 \quad 3 \quad 0 \\ | \quad | \quad | \quad | \quad | \end{array}
 \end{array}$$

to signify the application of relation (2.14).

We can define a linear map  $*$ :  $\mathcal{R}_n \rightarrow \mathcal{R}_n$  by swapping the diagrams of  $\mathcal{R}_n$  up-side-down. For example,

$$\left( \begin{array}{c} 0 \quad 1 \quad 3 \quad 2 \quad 2 \\ \diagup \quad \diagdown \\ | \quad | \end{array} \right)^* = \begin{array}{c} 3 \quad 0 \quad 2 \quad 1 \quad 2 \\ \diagdown \quad \diagup \\ | \quad | \end{array} .$$

It is obvious that  $*$  is an anti-isomorphism and it preserves the generators of  $\mathcal{R}_n$ .

Fix a weight  $\Lambda = \sum_{i \in I} a_i \Lambda_i$  with  $a_i \in \mathbb{N}$ . Let  $N_n^\Lambda(\mathcal{O})$  be the two-sided ideal of  $\mathcal{R}_n$  generated by the elements with form  $e(\mathbf{i}) y_1^{(\Lambda, \alpha_{i_1})}$ . We can now define the main object of study in this paper, the cyclotomic Khovanov-Lauda-Rouquier algebras, which were introduced by Khovanov and Lauda [13, Section 3.4].

**Definition 2.18.** *The cyclotomic Khovanov-Lauda-Rouquier algebras of weight  $\Lambda$  and type  $\Gamma_e$  is the algebra  $\mathcal{R}_n^\Lambda(\mathcal{O}) = \mathcal{R}_n(\mathcal{O}) / N_n^\Lambda(\mathcal{O})$ .*

Therefore, if we write  $e(\mathbf{i}) = \hat{e}(\mathbf{i}) + N_n^\Lambda(\mathcal{O})$ ,  $y_r = \hat{y}_r + N_n^\Lambda(\mathcal{O})$  and  $\psi_s = \hat{\psi}_s + N_n^\Lambda(\mathcal{O})$ , the algebra  $\mathcal{R}_n^\Lambda(\mathcal{O})$  is the unital  $\mathcal{O}$ -algebra generated by

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{e(\mathbf{i}) \mid \mathbf{i} \in I^n\}$$

subject to the relations (2.2)–(2.9) of  $\mathcal{R}_n(\mathcal{O})$  together with the additional relation

$$e(\mathbf{i})y_1^{(\Lambda, \alpha_{i_1})} = 0, \quad \text{for each } \mathbf{i} \in I^n. \quad (2.19)$$

## 2.2. The (graded) cellular algebras and the symmetric groups

Following Graham and Lehrer [7], we now introduce the graded cellular algebras. Reader may also refer to Hu-Mathas [9]. Let  $\mathcal{O}$  be a commutative ring with 1 and let  $A$  be a unital  $\mathcal{O}$ -algebra.

**Definition 2.20.** A *graded cell datum* for  $A$  is a triple  $(\Lambda, T, C, \deg)$  where  $\Lambda = (\Lambda, >)$  is a poset, either finite or infinite, and  $T(\lambda)$  is a finite set for each  $\lambda \in \Lambda$ ,  $\deg$  is a function from  $\coprod_\lambda T(\lambda)$  to  $\mathbb{Z}$ , and

$$C: \prod_{\lambda \in \Lambda} T(\lambda) \times T(\lambda) \longrightarrow A$$

is an injective map which sends  $(s, t)$  to  $a_{st}^\lambda$  such that:

- (a)  $\{a_{st}^\lambda \mid \lambda \in \Lambda, s, t \in T(\lambda)\}$  is an  $\mathcal{O}$ -free basis of  $A$ ;
- (b) for any  $r \in A$  and  $\mathbf{t} \in T(\lambda)$ , there exists scalars  $c_t^v(r)$  such that, for any  $\mathbf{s} \in T(\lambda)$ ,

$$a_{st}^\lambda \cdot r \equiv \sum_{v \in T(\lambda)} c_t^v(r) a_{sv}^\lambda \pmod{A^{>\lambda}}$$

where  $A^{>\lambda}$  is the  $\mathcal{O}$ -submodule of  $A$  spanned by  $\{a_{xy}^\mu \mid \mu > \lambda, x, y \in T(\mu)\}$ ;

(c) the  $\mathcal{O}$ -linear map  $*: A \longrightarrow A$  which sends  $a_{st}^\lambda$  to  $a_{ts}^\lambda$ , for all  $\lambda \in \Lambda$  and  $\mathbf{s}, \mathbf{t} \in T(\lambda)$ , is an anti-isomorphism of  $A$ .

(d) each basis element  $a_{st}^\lambda$  is homogeneous of degree  $\deg a_{st}^\lambda = \deg(\mathbf{s}) + \deg(\mathbf{t})$ , for  $\lambda \in \Lambda$  and all  $\mathbf{s}, \mathbf{t} \in T(\lambda)$ .

If a graded cell datum exists for  $A$  then  $A$  is a **graded cellular algebra**. Similarly, by forgetting the grading we can define a **cell datum** and hence a **cellular algebra**.

Suppose  $A$  is a graded cellular algebra with graded cell datum  $(\Lambda, T, C, \deg)$ . For any  $\lambda \in \Lambda$ , define  $A^{\geq \lambda}$  to be the  $\mathcal{O}$ -submodule of  $A$  spanned by

$$\{c_{st}^\mu \mid \mu \geq \lambda, \mathbf{s}, \mathbf{t} \in T(\mu)\}.$$

Then  $A^{>\lambda}$  is an ideal of  $A^{\geq \lambda}$  and hence  $A^{\geq \lambda}/A^{>\lambda}$  is a  $A$ -module. For any  $\mathbf{s} \in T(\lambda)$  we define  $C_s^\lambda$  to be the  $A$ -submodule of  $A^{\geq \lambda}/A^{>\lambda}$  with basis  $\{a_{st}^\lambda + A^{>\lambda} \mid \mathbf{t} \in T(\lambda)\}$ . By the cellularity of  $A$  we have  $C_s^\lambda \cong C_t^\lambda$  for any  $\mathbf{s}, \mathbf{t} \in T(\lambda)$ .

**Definition 2.21.** Suppose  $\lambda \in \mathcal{P}_n^\Lambda$ . Define the *cell module* of  $A$  to be  $C^\lambda = C_s^\lambda$  for any  $\mathbf{s} \in T(\lambda)$ , which has basis  $\{a_t^\lambda \mid \mathbf{t} \in T(\lambda)\}$  and for any  $r \in A$ ,

$$a_t^\lambda \cdot r = \sum_{u \in T(\lambda)} c_u^r a_u^\lambda$$

where  $c_u^r$  are determined by

$$a_{st}^\lambda \cdot r = \sum_{u \in T(\lambda)} c_u^r a_{su}^\lambda + A^{>\lambda}.$$

We can define a bilinear map  $\langle \cdot, \cdot \rangle: C^\lambda \times C^\lambda \longrightarrow \mathbb{Z}$  such that

$$\langle a_s^\lambda, a_t^\lambda \rangle a_{uv}^\lambda = a_{us}^\lambda a_{tv}^\lambda + A^{>\lambda}$$

and let  $\text{rad } C^\lambda = \{\mathbf{s} \in C^\lambda \mid \langle \mathbf{s}, \mathbf{t} \rangle = 0 \text{ for all } \mathbf{t} \in C^\lambda\}$ . The  $\text{rad } C^\lambda$  is a graded  $A$ -submodule of  $C^\lambda$ .

**Definition 2.22.** Suppose  $\lambda \in \mathcal{P}_n^\Lambda$ . Let  $D^\lambda = C^\lambda / \text{rad } C^\lambda$  as a graded  $A$ -module.

Exactly as in the ungraded case [7, Theorem 3.4] or [9, Theorem 2.10], we obtain the following:

**Theorem 2.23.** The set  $\{D^\lambda \langle k \rangle \mid \lambda \in \Lambda, D^\lambda \neq 0, k \in \mathbb{Z}\}$  is a complete set of pairwise non-isomorphic graded simple  $A$ -modules.

We give an example of graded cellular algebras here, which is called the cyclotomic Hecke algebras.

Let  $\mathbb{F}_p$  be a fixed field of characteristic  $p \geq 0$  with  $q \in \mathbb{F}_p^\times$ . Let  $e$  be the smallest positive integer such that  $1 + q + \dots + q^{e-1} = 0$  and setting  $e = 0$  if no such integer exists. Then define  $I = \mathbb{Z}/e\mathbb{Z}$  if  $e > 0$  and  $I = \mathbb{Z}$  if  $e = 0$ .

For  $n \geq 0$ , assume that  $q = 1$ . Let  $H_n$  be the **degenerate affine Hecke algebra**, working over  $\mathbb{F}_p$ . So  $H_n$  has generators

$$\{x_1, \dots, x_n\} \cup \{s_1, \dots, s_{n-1}\}$$

subject to the following relations

$$\begin{aligned} x_r x_s &= x_s x_r; \\ s_r x_{r+1} &= x_r s_r + 1, & s_r s_x &= x_s s_r & \text{if } s \neq r, r+1 \\ s_r^2 &= 1; \\ s_r s_{r+1} s_r &= s_{r+1} s_r s_{r+1}, & s_r s_t &= s_t s_r & \text{if } |r-t| > 1 \end{aligned}$$

Now we assume that  $q \neq 1$  and  $H_n$  be the **non-degenerate affine Hecke algebra** over  $\mathbb{F}_p$ . So  $H_n$  has generators

$$\{X_1^{\pm 1}, \dots, X_n^{\pm 1}\} \cup \{T_1, \dots, T_{n-1}\}$$

subject to the following relations

$$\begin{aligned} X_r^{\pm 1} X_s^{\pm 1} &= X_s^{\pm 1} X_r^{\pm 1}, & X_r X_r^{-1} &= 1; \\ T_r X_r T_r &= q X_{r+1}, & T_r X_s &= X_s T_r & \text{if } s \neq r, r+1; \\ T_r^2 &= (q-1)T_r + q; \\ T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1}, & T_r T_s &= T_s T_r & \text{if } |r-s| > 1. \end{aligned}$$

Then for any  $\Lambda \in P_+$ , we define

$$H_n^\Lambda = \begin{cases} H_n / \langle \prod_{i \in I} (X_i - q^i)^{(\Lambda, \alpha_i)} \rangle, & \text{if } q \neq 1, \\ H_n / \langle \prod_{i \in I} (X_i - i)^{(\Lambda, \alpha_i)} \rangle, & \text{if } q = 1. \end{cases} \quad (2.24)$$

and we call  $H_n^\Lambda$  the **degenerate cyclotomic Hecke algebra** if  $q = 1$  and **non-degenerate cyclotomic Hecke algebra** if  $q \neq 1$ .

By the definitions, degenerate and non-degenerate cyclotomic Hecke algebras are similar with some minor difference. In order to minimize their difference we define

$$q_i = \begin{cases} i, & \text{if } q = 1, \\ q^i, & \text{if } q \neq 1. \end{cases} \quad (2.25)$$

and use  $x_r$  instead of  $X_r$  when we don't have to distinguish which case we are working with. Hence we can re-write (2.24) as

$$H_n^\Lambda = H_n / \langle \prod_{i \in I} (x_i - q_i)^{(\Lambda, \alpha_i)} \rangle. \quad (2.26)$$

Murphy [22] gave a set of cellular basis for  $H_n^\Lambda$  which shows that  $H_n^\Lambda$  is a cellular algebra. Brundan and Kleshchev [3] proved the remarkable result that every  $H_n^\Lambda$  over  $\mathbb{F}_p$  is isomorphic to  $\mathcal{R}_n^\Lambda(\mathbb{F}_p)$  introduced in Definition 2.1, where in both algebras  $\Lambda$  and  $e$  are the same. Therefore when  $H_n^\Lambda$  is over a field it is a graded cellular algebra.

Let  $\mathfrak{S}_n$  be the symmetric group on  $\{1, 2, \dots, n\}$ . Then  $\mathfrak{S}_n$  is a Coxeter group and  $\{s_1, \dots, s_{n-1}\}$  is its standard set of Coxeter generators, where  $s_i = (i, i+1)$  for  $i = 1, 2, \dots, n-1$ . Suppose  $w$  is an element of  $\mathfrak{S}_n$  and  $w = s_{i_1} s_{i_2} \dots s_{i_m}$ . If  $m$  is minimal we say that  $w$  has **length**  $m$  and write  $l(w) = m$ . In this case we say  $s_{i_1} s_{i_2} \dots s_{i_m}$  is a **reduced expression** of  $w$ . In general an element of  $\mathfrak{S}_n$  has more than one reduced expressions. For example, we have  $w = s_1 s_2 s_1 = s_2 s_1 s_2$ . Nonetheless, all the reduced expression of an element have the same length.

In this paper we let  $\mathfrak{S}_n$  act on  $\{1, 2, \dots, n\}$  from right. For example,  $(i)s_i s_{i+1} = (i+1)s_{i+1} = i+2$ . The following result is well-known. See, for example, [20, Corollary 1.4].

**Proposition 2.27.** *Suppose that  $w \in \mathfrak{S}_n$ . For  $i = 1, 2, \dots, n-1$ ,*

$$l(ws_i) = \begin{cases} l(w) + 1, & \text{if } (i)w < (i+1)w, \\ l(w) - 1, & \text{if } (i)w > (i+1)w. \end{cases}$$

We recall the definition of the **Bruhat order**  $\leq$  on  $\mathfrak{S}_n$ . For  $u, w \in \mathfrak{S}_n$  define  $u \leq w$  if  $u = s_{r_{a_1}} s_{r_{a_2}} \dots s_{r_{a_b}}$  for some  $1 \leq a_1 < a_2 < \dots < a_b \leq m$ , where  $w = s_{r_1} s_{r_2} \dots s_{r_m}$  is a reduced expression for  $w$ .

### 2.3. Tableaux combinatorics

In this subsection we recall the combinatorics of (multi)partitions and (multi)tableaux that we will need in this paper.

Let  $n$  be a positive integer. A **composition** of  $n$  is an ordered sequence of nonnegative integers  $\lambda = (\lambda_1, \lambda_2, \dots)$  and  $|\lambda| = \sum_{i=1}^{\infty} \lambda_i = n$ . We say  $\lambda$  is a **partition** of  $n$  if  $\lambda = (\lambda_1, \lambda_2, \dots)$  is a composition and  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ . We can then identify  $\lambda$  with a sequence  $(\lambda_1, \dots, \lambda_k)$  whenever  $\lambda_i = 0$  for  $i > k$ .

As we now recall, there is a natural partial ordering on the set of compositions of  $n$ . Suppose  $\lambda = (\lambda_1, \lambda_2, \dots)$  and  $\mu = (\mu_1, \mu_2, \dots)$  are compositions of  $n$ . Then  $\lambda$  **dominates**  $\mu$ , and we write  $\lambda \triangleright \mu$ , if

$$\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$$

for any  $k$ . We write  $\lambda \triangleright \mu$  if  $\lambda \triangleright \mu$  and  $\lambda \neq \mu$ . The dominance ordering can be extended to a total ordering  $\geq$ , called the **lexicographic ordering**. We write  $\lambda > \mu$  if there exist some  $k$ , such that  $\lambda_i = \mu_i$  for all  $i < k$  and  $\lambda_k > \mu_k$ . Define  $\lambda \geq \mu$  if  $\lambda > \mu$  or  $\lambda = \mu$ . Then  $\lambda \triangleright \mu$  implies  $\lambda \geq \mu$ .

A **multicomposition** of  $n$  of **level**  $\ell$  is an ordered sequence  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  of compositions such that  $\sum_{i=1}^{\ell} |\lambda^{(i)}| = n$ . Similarly, a **multipartition** of level  $\ell$  is multicomposition  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  of  $n$  such that each  $\lambda^{(i)}$  is a partition. We will identify multicompositions and multipartitions of level 1 with compositions and partitions in the obvious way.

Let  $\mathcal{C}_n^\Lambda$  be the set of all multicomposition of  $n$  and  $\mathcal{P}_n^\Lambda$  be the set of all multipartitions of  $n$ . We can extend the dominance ordering to  $\mathcal{C}_n^\Lambda$  by defining  $\lambda \triangleright \mu$  if

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{j=1}^s \lambda_j^{(k)} \geq \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{j=1}^s \mu_j^{(k)}$$

for any  $1 \leq k \leq \ell$  and all  $s \geq 1$ . Again, we write  $\lambda \triangleright \mu$  if  $\lambda \triangleright \mu$  and  $\lambda \neq \mu$ . Similarly, we extend the lexicographic ordering  $\lambda > \mu$  and  $\lambda \geq \mu$  to  $\mathcal{C}_n^\Lambda$  in the obvious way.

The **Young diagram** of a multicomposition  $\lambda$  of level  $\ell$  is the set

$$[\lambda] = \{(r, c, l) \mid 1 \leq c \leq \lambda_r^{(l)}, r \geq 0 \text{ and } 1 \leq l \leq \ell\}$$

which we think of as an ordered  $\ell$ -tuple of the diagrams of the partitions  $\lambda^{(1)}, \dots, \lambda^{(\ell)}$ . The triple  $(r, c, l) \in [\lambda]$  is **node** of  $\lambda$  in row  $r$ , column  $c$  and component  $l$ . A  $\lambda$ -**tableau** is any bijection  $t: [\lambda] \rightarrow \{1, 2, \dots, n\}$ . We identify a  $\lambda$ -tableau  $t$  with a labeling of the diagram of  $\lambda$ . That is, we label the node  $(r, c, l) \in [\lambda]$  with the integer  $t(r, c, l)$ . For example,

$$\left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & & & \\ \hline \end{array} \middle| \begin{array}{|c|c|} \hline 9 & 10 \\ \hline 11 & 12 \\ \hline 13 & \\ \hline \end{array} \middle| \begin{array}{|c|c|c|c|} \hline 14 & 15 & 16 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right)$$

is a  $(4, 3, 1|2^2, 1|3)$ -tableaux. If  $t$  is a  $\lambda$ -tableau then the **shape** of  $t$  is the multicomposition  $\lambda$  and we write  $\text{Shape}(t) = \lambda$ . A  $\lambda$ -tableau  $t$  is **standard** if  $\lambda = \text{Shape}(t)$  is a multipartition and if, in each component, the entries increase along each row and down each column. More precisely, if  $(r, c, l) \in [\lambda]$  then  $t(r, c, l) < t(r+1, c, l)$  whenever  $(r+1, c, l) \in [\lambda]$  and  $t(r, c, l) < t(r, c+1, l)$  whenever  $(r, c+1, l) \in [\lambda]$ . Let  $\text{Std}(\lambda)$  be the set of all standard  $\lambda$ -tableaux and  $\text{Std}(> \lambda)$  be the set of all standard  $\mu$ -tableaux with  $\mu > \lambda$ . We can define  $\text{Std}(\geq \lambda)$  similarly. Note that if  $t$  is standard then so is  $t|_k$  for  $1 \leq k \leq n$ .

If  $t \in \text{Std}(\lambda)$  and  $1 \leq k \leq n$  define  $t|_k$  to be the subtableau of  $t$  obtained by removing all the nodes containing an entry greater than  $k$ . We define an analogue of the dominance ordering for standard tableaux by defining  $t \triangleright s$  if  $\text{Shape}(t|_k) \triangleright \text{Shape}(s|_k)$ , for  $1 \leq k \leq n$ . As with the dominance ordering, if  $t \triangleright s$  then we write  $s \trianglelefteq t$  and if  $s \neq t$  then write  $t \triangleright s$  and  $s \triangleleft t$ . We also define  $(s, t) \triangleright (u, v)$  if  $s \triangleright u$ ,  $t \triangleright v$  and  $(s, t) \neq (u, v)$ .

For any multicomposition  $\lambda$ , define  $t^\lambda$  to be the unique  $\lambda$ -tableau such that  $t^\lambda \triangleright t$  for all standard  $\lambda$ -tableau  $t$ . For example, if  $\lambda = (4, 3, 1|2^2, 1|3)$  then  $t^\lambda$  is the tableau displayed above.

The symmetric group acts on the set of all  $\lambda$ -tableaux. Let  $t$  be a  $\lambda$ -tableau, then  $t \cdot s_r$  is the tableau obtained by exchanging the entries  $r$  and  $r+1$  in  $t$ , i.e.  $(r)t^{-1} = (r+1)(t \cdot s_r)^{-1}$ ,  $(r+1)t^{-1} = (r)(t \cdot s_r)^{-1}$ , and  $(k)t^{-1} = (k)(t \cdot s_r)^{-1}$  for  $k \neq r, r+1$ . Then for each  $\lambda$ -tableau  $t$  let  $d(t)$  be the permutation in  $\mathfrak{S}_n$  such that  $t^\lambda \cdot d(t) = t$ .

Recall the Bruhat order  $\leq$  on  $\mathfrak{S}_n$  from subsection 1.1. The following result, which goes back to work of Ehresmann and James, is part of the folklore for these algebras. The proof for level 1 can be found from [20, Lemma 3.7]. The higher level cases follow easily.

**Lemma 2.28.** *Suppose  $\lambda \in \mathcal{P}_n^\Lambda$  and  $s$  and  $t$  are standard  $\lambda$ -tableaux. Then  $s \triangleright t$  if and only if  $d(s) \leq d(t)$ .*



Suppose  $\lambda$  is a multicomposition and  $\gamma = (r, c, l) \in [\lambda]$  and recall from subsection 1.1 that  $\kappa_\lambda = (\kappa_1, \kappa_2, \dots, \kappa_\ell)$  is a fixed multicharge of  $\Lambda$ . The **residue** of  $\gamma$  associate to  $\kappa_\lambda$  is

$$\text{res}(\gamma) \equiv r - c + \kappa_l \pmod{e}.$$

If  $\mathbf{t}$  is a standard  $\lambda$ -tableau and the **residue sequence** of  $\mathbf{t}$  is  $\text{res}(\mathbf{t}) = \mathbf{i}_t = (i_1, i_2, i_3, \dots, i_n)$ , where  $i_k = \text{res}(\gamma_k)$  and  $\gamma_k$  is the unique node in  $[\lambda]$  such that  $\mathbf{t}(\gamma_k) = k$ . In particular, we write  $\mathbf{i}_{t^i} = \mathbf{i}_\lambda$  and  $\text{res}_t(k) = \text{res}(\gamma_k)$ .

Recall that for each standard tableau  $\mathbf{t}$ , we can define a permutation  $d(\mathbf{t}) \in \mathfrak{S}_n$  such that  $\mathbf{t} = \mathbf{t}^\lambda \cdot d(\mathbf{t})$ . For each permutation we may have more than one reduced expression. Here we fix a choice of the reduced expression of  $d(\mathbf{t})$ .

For any standard  $\lambda$ -tableau  $\mathbf{t}$  and  $1 \leq i \leq n+1$ , define  $\lambda^{(i)} = \text{Shape}(\mathbf{t}|_{i-1})$  and  $\mathbf{t}^{(i)}$  to be a standard  $\lambda$ -tableau where  $\mathbf{t}^{(i)}|_{i-1} = \mathbf{t}^{(i)}$ , and  $\mathbf{t}^{(i)-1}(k) = \mathbf{t}^{-1}(k)$  for any  $i \leq k \leq n$ . In particular,  $\mathbf{t}^{(1)} = \mathbf{t}$  and  $\mathbf{t}^{(n+1)} = \mathbf{t}^\lambda$ . Therefore we have a series of standard  $\lambda$ -tableau

$$\mathbf{t}^\lambda = \mathbf{t}^{(n+1)}, \mathbf{t}^{(n)}, \mathbf{t}^{(n-1)}, \dots, \mathbf{t}^{(2)}, \mathbf{t}^{(1)} = \mathbf{t}.$$

Define  $w_i$  to be the unique permutation in  $\mathfrak{S}_n$  such that  $\mathbf{t}^{(i+1)}w_i = \mathbf{t}^{(i)}$ . For each  $w_i \neq 1$ , we can write  $w_i = s_{a_i}s_{a_i+1}s_{a_i+2} \dots s_{i-2}s_{i-1}$  for some  $a_i \leq i-1$ . Notice that

$$(k)(\mathbf{t}^{(i+1)})^{-1} = \begin{cases} (i)(\mathbf{t}^{(i)})^{-1}, & \text{if } k = a_i, \\ (k-1)(\mathbf{t}^{(i)})^{-1}, & \text{if } a_i < k \leq i, \\ (k)(\mathbf{t}^{(i)})^{-1}, & \text{otherwise,} \end{cases} \quad (2.29)$$

and  $l(w_i)$  is always greater than or equal to the length of the row containing  $i$  in  $\mathbf{t}^{(i+1)}$ . Also for each  $i$ , if  $\text{Shape}(\mathbf{t}^{(i)}|_{i-1}) = \lambda$ , then  $\mathbf{t}^{(i)}|_{i-1} = \mathbf{t}^\lambda$ .

**Example 2.30.** Suppose  $\mathbf{t}^{(11)} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 12 \\ \hline 5 & 6 & 7 & 11 & \\ \hline 8 & 9 & 10 & 13 & \\ \hline 14 & 15 & & & \\ \hline \end{array}$  and  $\mathbf{t}^{(10)} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 10 & 12 \\ \hline 4 & 5 & 6 & 11 & \\ \hline 7 & 8 & 9 & 13 & \\ \hline 14 & 15 & & & \\ \hline \end{array}$ . Therefore we have  $w_{10} = s_4s_5s_6s_7s_8s_9$  such that  $\mathbf{t}^{(11)} \cdot w_{10} = \mathbf{t}^{(10)}$ .

Notice that in this case,  $i = 10$  and  $a_{10} = 4$ . So

$$(k)(\mathbf{t}^{(11)})^{-1} = \begin{cases} (10)(\mathbf{t}^{(10)})^{-1}, & \text{if } k = a_i = 4, \\ (k-1)(\mathbf{t}^{(10)})^{-1}, & \text{if } 4 = a_i < k \leq i = 10, \\ (k)(\mathbf{t}^{(10)})^{-1}, & \text{otherwise.} \end{cases}$$

$$\text{Furthermore, } \mathbf{t}^{(11)}|_{10} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & 9 & 10 & \\ \hline \end{array} = \mathbf{t}^{(4,3,3)} \text{ and } \mathbf{t}^{(10)}|_9 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} = \mathbf{t}^{(3,3,3)}.$$

One can see that  $d(\mathbf{t}) = w_n w_{n-1} \dots w_2 w_1$  and define it to be the **standard expression** of  $d(\mathbf{t})$ . By Proposition 2.27 one can verify that standard expression is a reduced expression of  $d(\mathbf{t})$ . In the rest of this paper, we fix  $d(\mathbf{t})$  to be its standard expression.

**Lemma 2.31.** Suppose  $\mathbf{t}$  is a standard  $\lambda$ -tableau and  $d(\mathbf{t}) = s_{r_1} \dots s_{r_m}$  is the standard expression. For any  $1 \leq k \leq m$ , define  $\mathbf{s} = \mathbf{t}^\lambda \cdot s_{r_1} s_{r_2} \dots s_{r_k}$ . Then  $\mathbf{s}$  is a standard  $\lambda$ -tableau.

*Proof.* The proof is trivial by the definition of the standard expression.  $\square$

**Example 2.32.** Suppose  $\mathbf{t} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 6 & 7 \\ \hline 3 & 5 & & & \\ \hline \end{array}$ . Then we have  $d(\mathbf{t}) = s_5 s_6 \cdot s_4 s_5 \cdot s_3$ . Then

$$\begin{aligned} \mathbf{t}^\lambda \cdot s_5 &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 6 \\ \hline 5 & 7 & & & \\ \hline \end{array}, \\ \mathbf{t}^\lambda \cdot s_5 s_6 &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 7 \\ \hline 5 & 6 & & & \\ \hline \end{array}, \\ \mathbf{t}^\lambda \cdot s_5 s_6 s_4 &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 7 \\ \hline 4 & 6 & & & \\ \hline \end{array}, \\ \mathbf{t}^\lambda \cdot s_5 s_6 s_4 s_5 &= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & & & \\ \hline \end{array}, \end{aligned}$$

and the above tableaux are all standard.

## 2.4. Graded cellular basis of KLR algebras over a field

Suppose  $\mathcal{O}$  is a field. Hu and Mathas [9, Theorem 5.8] constructed a graded cellular basis of  $\mathcal{R}_n^\Lambda(\mathcal{O})$ . Here we give an equivalent definition of their basis. For any multicomposition  $\lambda$ , recall  $\mathbf{t}^\lambda$  to be the unique standard  $\lambda$ -tableau such that  $\mathbf{t}^\lambda \geq \mathbf{t}$  for all standard  $\lambda$ -tableau  $\mathbf{t}$ , and  $\mathbf{i}_\lambda$  is the residue sequence of  $\mathbf{t}^\lambda$ . We define  $\hat{e}_\lambda = \hat{e}(\mathbf{i}_\lambda)$ .

Suppose  $\lambda$  is a multicomposition. A node  $(r, c, l)$  is an **addable node** of  $\lambda$  if  $(r, c, l) \notin [\lambda]$  and  $[\lambda] \cup \{(r, c, l)\}$  is the Young diagram of a multipartition. Similarly, a node  $(r, c, l)$  is a **removable node** of  $\lambda$  if  $(r, c, l) \in [\lambda]$  and

$[\lambda] \setminus \{(r, c, l)\}$  is the Young diagram of a multipartition. Given two nodes  $\alpha = (r, c, l)$  and  $\beta = (s, t, m)$  then  $\alpha$  is **below**  $\beta$  if either  $l > m$ , or  $l = m$  and  $r > s$ .

Suppose that  $\mathbf{s} \in \text{Std}(\lambda)$ . Let  $\mathcal{A}_{\mathbf{s}}(k)$  be the set of addable nodes of the multicomposition  $\text{Shape}(\mathbf{s}|_k)$  which are below  $\mathbf{s}^{-1}(k)$  and let

$$\mathcal{A}_{\mathbf{s}}^{\Lambda}(k) = \{\alpha \in \mathcal{A}_{\mathbf{s}}(k) \mid \text{res}(\alpha) = \text{res}_{\mathbf{t}}(k)\}.$$

Similarly as in [9, Definition 4.12], define

$$\hat{y}_{\lambda} = \prod_{k=1}^n \hat{y}_k^{|\mathcal{A}_{\mathbf{t}}^{\Lambda}(k)|} \in \mathcal{R}_n(\mathcal{O}).$$

For example, if  $\lambda = (3, 1|4^2, 2|5, 1)$ ,  $e = 4$  and  $\Lambda = 3\Lambda_0$  then

$$\mathbf{t}^{\lambda} = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \middle| \begin{array}{|c|c|c|c|} \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & & \\ \hline \end{array} \middle| \begin{array}{|c|c|c|c|c|c|} \hline 15 & 16 & 17 & 18 & 19 \\ \hline 20 & & & & \\ \hline \end{array} \right)$$

and  $\hat{y}_{\lambda} = \hat{y}_1^2 \hat{y}_5 \hat{y}_8 \hat{y}_{10} \hat{y}_{12} \hat{y}_{18}$ . Therefore,

$$\hat{e}_{\lambda} \hat{y}_{\lambda} = \hat{e}(0123012330122012303) \hat{y}_1^2 \hat{y}_5 \hat{y}_8 \hat{y}_{10} \hat{y}_{12} \hat{y}_{18}.$$

We define a particular kind of element in  $\mathcal{R}_n(\mathcal{O})$ . Suppose  $w \in \mathfrak{S}_n$  has length  $\ell$  and  $s_{i_1} s_{i_2} \dots s_{i_{\ell}}$  is a reduced expression for  $w$  in  $\mathfrak{S}_n$ . Recall that  $\mathcal{R}_n(\mathcal{O})$  has a unique anti-isomorphism  $*$  which fixes all of the KLR generators. Define

$$\hat{\psi}_w = \hat{\psi}_{i_1} \hat{\psi}_{i_2} \dots \hat{\psi}_{i_{\ell}} \in \mathcal{R}_n(\mathcal{O}) \quad \text{and} \quad \hat{\psi}_w^* = \hat{\psi}_{i_{\ell}} \hat{\psi}_{i_{\ell-1}} \dots \hat{\psi}_{i_2} \hat{\psi}_{i_1} \in \mathcal{R}_n(\mathcal{O}).$$

Notice that  $\hat{\psi}_w$  and  $\hat{\psi}_w^*$  depend on the choice of the reduced expression of  $w$ , even though in  $\mathfrak{S}_n$  all reduced expressions of  $w$  are the same. For example,  $s_1 s_2 s_1$  and  $s_2 s_1 s_2$  are equal to the same element of  $\mathfrak{S}_n$ , but in general  $\hat{\psi}_1 \hat{\psi}_2 \hat{\psi}_1 \neq \hat{\psi}_2 \hat{\psi}_1 \hat{\psi}_2$  in  $\mathcal{R}_n(\mathcal{O})$ . Define  $l(\hat{\psi}_w) = l(\hat{\psi}_w^*) = l(w)$  for any standard tableau  $\mathbf{t}$ . Similarly we can define

$$\psi_w = \psi_{i_1} \psi_{i_2} \dots \psi_{i_{\ell}} \in \mathcal{R}_n^{\Lambda}(\mathcal{O}) \quad \text{and} \quad \psi_w^* = \psi_{i_{\ell}} \psi_{i_{\ell-1}} \dots \psi_{i_2} \psi_{i_1} \in \mathcal{R}_n^{\Lambda}(\mathcal{O})$$

and  $\psi_w$  and  $\psi_w^*$  depends on the choice of reduced expressions of  $w$  as well.

Suppose  $l(d(\mathbf{t})) = \ell$  and  $d(\mathbf{t}) = s_{i_1} s_{i_2} \dots s_{i_{\ell}}$  is the standard expression of  $d(\mathbf{t})$  where  $\mathbf{t}^{\lambda} \cdot d(\mathbf{t}) = \mathbf{t}$ . Define  $\hat{\psi}_{d(\mathbf{t})} = \hat{\psi}_{i_1} \hat{\psi}_{i_2} \dots \hat{\psi}_{i_{\ell}}$  and  $\hat{\psi}_{d(\mathbf{t})}^* = \hat{\psi}_{i_{\ell}} \hat{\psi}_{i_{\ell-1}} \dots \hat{\psi}_{i_2} \hat{\psi}_{i_1}$ .

**Definition 2.33.** Suppose  $\Lambda \in P_+$ ,  $\lambda \in \mathcal{P}_n^{\Lambda}$  and  $\mathbf{s}, \mathbf{t}$  are two standard  $\lambda$ -tableaux. We define

$$\hat{\psi}_{\mathbf{st}}^{\mathcal{O}} = \hat{\psi}_{d(\mathbf{s})}^* \hat{e}_{\lambda} \hat{y}_{\lambda} \hat{\psi}_{d(\mathbf{t})} \in \mathcal{R}_n(\mathcal{O}),$$

and hence

$$\psi_{\mathbf{st}}^{\mathcal{O}} = \hat{\psi}_{\mathbf{st}}^{\mathcal{O}} + N_n^{\Lambda} \in \mathcal{R}_n^{\Lambda}(\mathcal{O}).$$

**Remark 2.34.** Notice that Hu and Mathas [9, Definition 5.1] defined  $\psi_{\mathbf{st}}^{\mathcal{O}}$  differently. Actually if we define  $e_{\lambda}, y_{\lambda}$  and  $\psi_w$  in  $\mathcal{R}_n^{\Lambda}(\mathcal{O})$  as analogues of  $\hat{e}_{\lambda}, \hat{y}_{\lambda}$  and  $\hat{\psi}_w$ , and define  $\psi_{\mathbf{st}}^{\mathcal{O}} = \psi_{d(\mathbf{s})}^* e_{\lambda} y_{\lambda} \psi_{d(\mathbf{t})}$  for  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$ , it is equivalent to Definition 2.33. We define  $\psi_{\mathbf{st}}^{\mathcal{O}}$  as in Definition 2.33 because we need to work in  $\mathcal{R}_n(\mathcal{O})$  later.

**Remark 2.35.** By construction, then this  $\psi_{\mathbf{st}}^{\mathcal{O}}$  is well defined as an element of  $\mathcal{R}_n^{\Lambda}(\mathcal{O})$  for any ring  $\mathcal{O}$ . Many of the calculations in this paper depend heavily on the choice of  $\mathcal{O}$  so we write  $\psi_{\mathbf{st}}^{\mathcal{O}}$  to emphasize that  $\psi_{\mathbf{st}}^{\mathcal{O}}$  is an element of  $\mathcal{R}_n^{\Lambda}(\mathcal{O})$ . Most of the time, however, we will work in  $\mathcal{R}_n^{\Lambda}(\mathbb{Z})$  so for convenience we set  $\psi_{\mathbf{st}} = \psi_{\mathbf{st}}^{\mathbb{Z}}$ .

**Lemma 2.36** (Hu and Mathas [9, Lemma 5.2] [10, Corollary 3.11, 3.12]). Suppose  $\mathcal{O}$  is a field and  $\mathbf{s}$  and  $\mathbf{t}$  are standard  $\lambda$ -tableaux and  $1 \leq r \leq n$ ,

$$\psi_{\mathbf{st}} \psi_r = \begin{cases} \sum_{(u,v) \triangleright (\mathbf{s}, \mathbf{t})} c_{uv} \psi_{uv}, & \text{if } \mathbf{t} \cdot s_r \text{ is not standard} \\ & \text{or } d(\mathbf{t}) \cdot s_r \text{ is not reduced,} \\ \psi_{\mathbf{sv}} + \sum_{(u,v) \triangleright (\mathbf{s}, \mathbf{t})} c_{uv} \psi_{uv}, & \text{if } \mathbf{v} = \mathbf{t} \cdot s_r \text{ standard and } d(\mathbf{t}) \cdot s_r = d(\mathbf{v}). \end{cases}$$

for  $c_{uv} \in \mathcal{O}$ , and  $c_{uv} \neq 0$  only if  $\text{res}(\mathbf{s}) = \text{res}(\mathbf{u})$  and  $\text{res}(\mathbf{t} \cdot s_r) = \text{res}(\mathbf{v})$ . Similarly, we have

$$\psi_{\mathbf{st}}^{\mathcal{O}} y_r = \sum_{(u,v) \triangleright (\mathbf{s}, \mathbf{t})} c_{uv} \psi_{uv}^{\mathcal{O}}$$

for  $c_{uv} \in \mathcal{O}$ , and  $c_{uv} \neq 0$  only if  $\text{res}(\mathbf{s}) = \text{res}(\mathbf{u})$  and  $\text{res}(\mathbf{t}) = \text{res}(\mathbf{v})$ .

**Theorem 2.37** (Hu and Mathas [9, Theorem 5.14]). Suppose  $\mathcal{O}$  is an integral domain and that either  $e = 0$ ,  $e$  is a prime or  $e$  is a non-zero non-prime integer such that  $e \cdot 1_{\mathcal{O}}$  is invertible in  $\mathcal{O}$ . Then

$$\{\psi_{\mathbf{st}}^{\mathcal{O}} \mid \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n^{\Lambda}\}$$

is a graded cellular basis of  $\mathcal{R}_n^{\Lambda}(\mathcal{O})$ . In particular,  $\mathcal{R}_n^{\Lambda}(\mathcal{O})$  is free as an  $\mathcal{O}$ -module of rank  $\ell^n n!$ .

The main purpose of this paper is to prove that  $\mathcal{R}_n^\Lambda(\mathbb{Z})$  is free of rank  $\ell^n n!$ . To do this we will show that  $\{\psi_{st}^\mathbb{Z} \mid s, t \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n^\Lambda\}$  is a homogeneous basis of  $\mathcal{R}_n^\Lambda(\mathbb{Z})$ .

We define some notation for future use.

**Definition 2.38.** Suppose  $\lambda$  is a multipartition of  $\mathcal{P}_n^\Lambda$ . Define:

$$\begin{aligned} R_n^\Lambda &= \langle \psi_{st} \mid s, t \in \text{Std}(\mu) \text{ for } \mu \in \mathcal{P}_n^\Lambda \rangle_{\mathbb{Z}}, \\ R_n^{\geq \lambda} &= \langle \psi_{st} \mid s, t \in \text{Std}(\mu) \text{ and } \mu \geq \lambda \text{ for } \mu \in \mathcal{P}_n^\Lambda \rangle_{\mathbb{Z}}, \\ R_n^{> \lambda} &= \langle \psi_{st} \mid s, t \in \text{Std}(\mu) \text{ and } \mu > \lambda \text{ for } \mu \in \mathcal{P}_n^\Lambda \rangle_{\mathbb{Z}}. \end{aligned}$$

where  $R_n^{> \lambda} \subseteq R_n^{\geq \lambda} \subseteq R_n^\Lambda \subseteq \mathcal{R}_n^\Lambda(\mathbb{Z})$

This subsection closes with an important Proposition:

Consider the quiver Hecke algebra  $\mathcal{R}_n(\mathbb{Q})$  defined over the rational field  $\mathbb{Q}$ . We have  $\mathcal{R}_n(\mathbb{Q}) \cong \mathcal{R}_n(\mathbb{Z}) \otimes \mathbb{Q}$  and we can define a linear map  $f : \mathcal{R}_n(\mathbb{Z}) \rightarrow \mathcal{R}_n(\mathbb{Q})$  by sending  $x \in \mathcal{R}_n(\mathbb{Z})$  to  $x \otimes 1$ .

**Lemma 2.39.** The linear map  $f : \mathcal{R}_n(\mathbb{Z}) \rightarrow \mathcal{R}_n(\mathbb{Q})$  is an injection.

*Proof.* In [13][12], Khovanov and Lauda constructed a basis of  $\mathcal{R}_n(\mathcal{O})$

$$\{ \hat{e}(\mathbf{i}) \hat{y}_1^{\ell_1} \hat{y}_2^{\ell_2} \dots \hat{y}_n^{\ell_n} \hat{\psi}_w \mid \mathbf{i} \in I^n, w \in \mathfrak{S}_n, \ell_1, \ell_2, \dots, \ell_n \geq 0 \} \quad (2.40)$$

for any ring  $\mathcal{O}$ . Hence that  $\mathcal{R}_n(\mathbb{Z})$  is free over  $\mathbb{Z}$ . The Lemma follows immediately.  $\square$

From the definitions, it is evident that  $f(N_n^\Lambda(\mathbb{Z})) \subseteq N_n^\Lambda(\mathbb{Q})$ . Hence,  $f$  induces a homomorphism,

$$f : \mathcal{R}_n^\Lambda(\mathbb{Z}) \rightarrow \mathcal{R}_n^\Lambda(\mathbb{Q}); x + N_n^\Lambda(\mathbb{Z}) \mapsto f(x) + N_n^\Lambda(\mathbb{Q}),$$

which by abuse of notation we also denote by  $f$ . In particular, observe that  $f(\psi_{st}^\mathbb{Z}) = \psi_{st}^\mathbb{Q}$ . The main Theorem of this paper is equivalently to prove that  $f : \mathcal{R}_n^\Lambda(\mathbb{Z}) \rightarrow \mathcal{R}_n^\Lambda(\mathbb{Q})$  is an injection.

We then introduce an important special case where we already know that  $f$  is injective.

**Proposition 2.41.** The homomorphism  $f : \mathcal{R}_n^\Lambda(\mathbb{Z}) \rightarrow \mathcal{R}_n^\Lambda(\mathbb{Q})$  restricts to an injective map from  $R_n^\Lambda$  to  $\mathcal{R}_n^\Lambda(\mathbb{Q})$ .

*Proof.* As we have already noted above,  $f(\psi_{st}^\mathbb{Z}) = \psi_{st}^\mathbb{Q}$  for all  $s, t \in \text{Std}(\lambda)$  and  $\lambda \in \mathcal{P}_n^\Lambda$ . Hence, Theorem 2.37 implies the result.  $\square$

**Corollary 2.42.** The elements  $\{\psi_{st}^\mathbb{Z} \mid s, t \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n^\Lambda\}$  are a linearly independent subset of  $\mathcal{R}_n^\Lambda(\mathbb{Z})$ .

**Remark 2.43.** Proposition 2.41 is quite crucial. In this paper we prove that  $\psi_{st}^\mathbb{Z} \cdot \psi_r \in R_n^\Lambda$  whenever  $d(t) \cdot s_r$  is not reduced or  $t \cdot s_r$  is not standard in  $\mathcal{R}_n^\Lambda(\mathbb{Z})$ . We can only have

$$\psi_{st}^\mathbb{Z} \cdot \psi_r = \sum_{u,v} c_{uv}^\mathbb{Z} \psi_{uv}^\mathbb{Z}.$$

In  $\mathcal{R}_n^\Lambda(\mathbb{Q})$ , however, by Lemma 2.36, under these conditions we have

$$\psi_{st}^\mathbb{Q} \cdot \psi_r = \sum_{(u,v) \triangleright (s,t)} c_{uv}^\mathbb{Q} \psi_{uv}^\mathbb{Q}$$

for some  $c_{uv}^\mathbb{Q} \in \mathbb{Q}$ , where  $(u, v) \triangleright (s, t)$  if  $u \geq s$ ,  $v \geq t$  and  $(u, v) \neq (s, t)$ . Therefore,  $c_{uv}^\mathbb{Q} = c_{uv}^\mathbb{Z}$  by Proposition 2.41 and we see that  $c_{uv}^\mathbb{Z} \neq 0$  only if  $(u, v) \triangleright (s, t)$ . In such case we have much more information about  $u$  and  $v$  with  $c_{uv}^\mathbb{Z} \neq 0$ . Similar remarks apply to the products  $\psi_{st}^\mathbb{Z} \cdot \psi_r$ .

### 3. Integral Basis Theorem I

In the next two sections we will prove that  $\mathcal{R}_n^\Lambda$  is  $\mathbb{Z}$ -free by showing that  $\{\psi_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n^\Lambda\}$  spans  $\mathcal{R}_n^\Lambda$  over  $\mathbb{Z}$ . Then by Corollary 2.42 it is a basis of  $\mathcal{R}_n^\Lambda$  over  $\mathbb{Z}$ .

In the rest of this paper we write  $\mathcal{R}_n(\mathbb{Z})$  as  $\mathcal{R}_n$  and  $\mathcal{R}_n^\Lambda(\mathbb{Z})$  as  $\mathcal{R}_n^\Lambda$ . Fix a weight  $\Lambda$ , a multicharge  $\kappa_\Lambda = (\kappa_1, \dots, \kappa_l)$  corresponding to  $\Lambda$  and an integer  $e > 2$ . In this and the next section we mainly work with the algebra  $\mathcal{R}_n^\Lambda$ .

### 3.1. The base step of the induction

In this subsection we set up the notations and inductive machinery that we use in the next two sections to prove our main theorem. We then consider the base case of our induction which is when  $\lambda = (n|\emptyset|\dots|\emptyset)$ . Finally we develop some technical Lemmas which will be useful later.

**Definition 3.1.** Suppose that  $\lambda$  is a multipartition of  $n$ . Let  $\lambda^+$  be the multicomposition of  $n+1$  obtained by adding a node at the end of the last non-empty row of  $\lambda$ , and  $\lambda_- = \lambda|_{n-1}$  be the multipartition of  $n-1$  obtained by removing the last node from  $\lambda$ .

For example, if  $\lambda = (4, 3|3, 3)$  then  $\lambda^+ = (4, 3|3, 4)$  and  $\lambda_- = (4, 3|3, 2)$ . Notice that in general,  $\lambda^+$  will be a multicomposition rather than a multipartition.

For  $k \in I$  and  $\lambda \in \mathcal{P}_n^\Lambda$ , define  $\mathcal{A}_{\mathbf{t}^\lambda}^k = \{\alpha \in \mathcal{A}_{\mathbf{t}^\lambda}(n) \mid \text{res}(\alpha) = k\}$ . Recall  $\mathbf{i}_\lambda = \text{res}(\mathbf{t}^\lambda)$  and  $e_\lambda = e(\mathbf{i}_\lambda)$  from subsection 2.4.

**Definition 3.2.** Suppose that  $\lambda \in \mathcal{P}_n^\Lambda$  and  $k \in I$ . Define the integer  $b_k^\lambda$  by

$$b_k^\lambda = \begin{cases} |\mathcal{A}_{\mathbf{t}^\lambda}^k| + 1, & \text{if } \lambda^+ \text{ is a multipartition and } i_n + 1 = k, \\ |\mathcal{A}_{\mathbf{t}^\lambda}^k|, & \text{otherwise.} \end{cases}$$

If  $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$  and  $k \in I$  then define  $\mathbf{i} \vee k = (i_1, i_2, \dots, i_n, k) \in I^{n+1}$ .

**Lemma 3.3.** Suppose that  $\lambda \in \mathcal{P}_n^\Lambda$  and  $k \in I$ . Then for each integer  $b$  with  $0 \leq b < b_k^\lambda$ , there exists a multipartition  $\nu = \nu(b)$  such that  $e_\nu y_\nu = e(\mathbf{i}_\lambda \vee k) y_\lambda y_{n+1}^b$ .

*Proof.* The definitions of  $\lambda$  and  $b_k^\lambda$  ensure that there are  $b_k^\lambda$  addable nodes of residue  $k$  below  $(\mathbf{t}^\lambda)^{-1}(n)$ . Suppose those nodes are  $(r_1, c_1, l_1), (r_2, c_2, l_2), \dots, (r_{b_k^\lambda}, c_{b_k^\lambda}, l_{b_k^\lambda})$ , where  $l_1 \geq l_2 \geq l_3 \geq \dots \geq l_{b_k^\lambda}$ , and if  $l_i = l_{i+1}$  then  $r_i \geq r_{i+1}$ . In another word,  $(r_i, c_i, l_i)$  is a node below  $(r_{i+1}, c_{i+1}, l_{i+1})$ .

For any  $b$  with  $0 \leq b < b_k^\lambda$ , we define  $\nu$  to be the multipartition obtained by adding the node  $(r_{b+1}, c_{b+1}, l_{b+1})$  on to  $\lambda$ . Then  $y_\nu = y_\lambda y_{n+1}^b$  and  $e_\nu = e(\mathbf{i}_\lambda \vee k) = e(\mathbf{i} \vee k)$ . This completes the proof.  $\square$

**Example 3.4.** Suppose that  $\lambda = (4, 3|2, 1|\emptyset|\emptyset)$  with  $e = 4$  and  $\kappa_\Lambda = (0, 0, 2, 1)$ . Then  $e(\mathbf{i}_\lambda) = e(0123301013)$  and  $y_\lambda = y_1 y_2 y_3 y_4 y_6 y_7 y_9$ . Then  $b_0^\lambda = 1, b_1^\lambda = 1, b_2^\lambda = 2$  and  $b_3^\lambda = 0$  and the proof of Lemma 3.3 shows that:

$$\begin{aligned} e(01233010130) y_1 y_2 y_3 y_4 y_6 y_7 y_9 &= e_{\mu_1} y_{\mu_1}, \\ e(01233010131) y_1 y_2 y_3 y_4 y_6 y_7 y_9 &= e_{\mu_2} y_{\mu_2}, \\ e(01233010132) y_1 y_2 y_3 y_4 y_6 y_7 y_9 &= e_{\mu_3} y_{\mu_3}, \\ e(01233010132) y_1 y_2 y_3 y_4 y_6 y_7 y_9 y_{11} &= e_{\mu_4} y_{\mu_4}, \end{aligned}$$

where  $\mu_1 = (4, 3|2, 2|\emptyset|\emptyset), \mu_2 = (4, 3|2, 1|\emptyset|1), \mu_3 = (4, 3|2, 1|1|\emptyset)$  and  $\mu_4 = (4, 3|2, 1, 1|\emptyset|\emptyset)$ .

**Definition 3.5.** Let  $\mathcal{P}^\Lambda = \cup_{n \geq 0} \mathcal{P}_n^\Lambda$ . Define three sets  $\mathcal{P}_I^\Lambda, \mathcal{P}_y^\Lambda$  and  $\mathcal{P}_\psi^\Lambda$  of multipartitions by:

$$\begin{aligned} \mathcal{P}_I^\Lambda &= \{\lambda \in \mathcal{P}^\Lambda \mid |\lambda| = n \text{ and } e(\mathbf{i}_\lambda \vee k) y_\lambda y_n^{b_k^\lambda} \in R_n^{\Lambda, \lambda} \text{ for all } k \in I\}, \\ \mathcal{P}_y^\Lambda &= \{\lambda \in \mathcal{P}^\Lambda \mid |\lambda| = n \text{ and } \psi_{\mathbf{s}\mathbf{t}} y_r \in R_n^\Lambda \text{ whenever } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ and } 1 \leq r \leq n\}, \\ \mathcal{P}_\psi^\Lambda &= \{\lambda \in \mathcal{P}^\Lambda \mid |\lambda| = n \text{ and } \psi_{\mathbf{s}\mathbf{t}} \psi_r \in R_n^\Lambda \text{ whenever } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ and } 1 \leq r < n\}. \end{aligned}$$

**Remark 3.6.** Notice that if for some  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$  and  $1 \leq r \leq n$  we have  $\psi_{\mathbf{s}\mathbf{t}} y_r \in R_n^\Lambda$ , then  $y_r \psi_{\mathbf{s}\mathbf{t}} \in R_n^\Lambda$  as well. Similar property holds for  $\psi_{\mathbf{s}\mathbf{t}} \psi_r$ . Therefore we can write

$$\begin{aligned} \mathcal{P}_y^\Lambda &= \{\lambda \in \mathcal{P}^\Lambda \mid |\lambda| = n \text{ and } y_r \psi_{\mathbf{s}\mathbf{t}} \in R_n^\Lambda \text{ whenever } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ and } 1 \leq r \leq n\}, \\ \mathcal{P}_\psi^\Lambda &= \{\lambda \in \mathcal{P}^\Lambda \mid |\lambda| = n \text{ and } \psi_r \psi_{\mathbf{s}\mathbf{t}} \in R_n^\Lambda \text{ whenever } \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ and } 1 \leq r < n\} \end{aligned}$$

as well.

By Proposition 2.41 if one of  $e(\mathbf{i}_\nu) y_\nu, \psi_{\mathbf{s}\mathbf{t}} y_r$  or  $\psi_{\mathbf{s}\mathbf{t}} \psi_r$  belongs to  $R_n^\Lambda$  then it can be written in a unique way as an (integral) linear combination of the  $\psi$ -basis elements. In particular, these linear combinations must satisfy the restrictions imposed by Lemma 2.36.

We define a total ordering on  $\mathcal{P}^\Lambda$  extended by lexicographic ordering. Suppose  $\lambda$  and  $\mu$  are two multipartitions, not necessarily of the same integer. Define  $\mu < \lambda$  if  $|\mu| < |\lambda|$ , or  $|\mu| = |\lambda|$  and  $l(\mu) < l(\lambda)$ , or  $|\mu| = |\lambda|, l(\mu) = l(\lambda)$  and  $\lambda < \mu$ .

**Definition 3.7.** Define  $\mathcal{S}_n^\Lambda = \{\lambda \in \mathcal{P}_n^\Lambda \mid \mu \in \mathcal{P}_I^{\Lambda'} \cap \mathcal{P}_y^{\Lambda'} \cap \mathcal{P}_\psi^{\Lambda'} \text{ whenever } \mu \in \mathcal{P}_m^{\Lambda'} \text{ and } \mu < \lambda\}$

We note that  $\mathcal{P}_n^\Lambda \subseteq \mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$  implies  $R_n^\Lambda = \mathcal{R}_n^\Lambda$ . Recall that  $R_n^\Lambda$  is the  $\mathbb{Z}$ -vector space spanned by  $\{\psi_{st}\}$ . Hence  $\mathcal{R}_n^\Lambda$  is a  $\mathbb{Z}$ -span of  $\{\psi_{st}\}$ . The main goal of this and the next sections is to prove that  $\mathcal{P}_n^\Lambda \subseteq \mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$  and we will use induction to prove the argument.

Now we can state the main result of this section.

**Theorem 3.8.** *Suppose  $\lambda \in \mathcal{S}_n^\Lambda$ . Then we have  $\lambda \in \mathcal{P}_I^\Lambda$ .*

As we mentioned before we are going to apply induction on  $\lambda$  to prove the main Theorem. Lemma 3.9, Corollary 3.10 and Corollary 3.11 give the base case of the induction. Recall that  $e \neq 2$ .

**Lemma 3.9.** *Suppose that  $n \geq 1$  and  $\lambda = (n|0| \dots |0) \in \mathcal{P}_n^\Lambda$ . Then  $e(\mathbf{i}_\lambda \vee k)y_{\lambda_-}y_n^{b_k^\lambda} \in R_n^{\geq \lambda}$  for any  $k \in I$ .*

*Proof.* As  $\lambda$  is the maximal element of  $\mathcal{P}_n^\Lambda$ ,  $R_n^{\geq \lambda} = \{0\}$ . Therefore the Lemma is equivalent to the claim that  $e(\mathbf{i}_\lambda \vee k)y_{\lambda_-}y_n^{b_k^\lambda} = 0$ . We prove this by induction on  $n$ .

If  $n = 1$  then it is easy to see that  $b_k^\lambda = (\Lambda, \alpha_{i_1})$ . Therefore,  $e(\mathbf{i}_\lambda \vee k)y_{\lambda_-}y_n^{b_k^\lambda} = e(k)y_1^{(\Lambda, \alpha_{i_1})} = 0$  by (2.19).

Suppose now that the Lemma holds for any  $n' < n$ . Notice that for any  $m \geq 1$ , set  $\gamma = (m|0| \dots |0)$  then  $|\mathcal{A}_\gamma^k|$  is independent to the value of  $m$ . For the rest of the proof we set  $a_k = |\mathcal{A}_{\lambda_-}^k|$ .

In order to simplify the notations, for the rest of the proof we will omit  $i_1 i_2 \dots i_{n-3}$  and simply write  $e(\mathbf{i}) = e(i_{n-2}, i_{n-1}, i_n)$ . We will also suppress  $y_\nu$ , where  $\nu = \lambda|_{n-3}$ .

We consider four cases separately, depending on the value of  $k$ .

**Case 3.9a:**  $k = i_{n-1}$ . Then  $e(\mathbf{i}_\lambda \vee k)y_{\lambda_-}y_n^{b_k^\lambda} = e(\mathbf{i}_{\lambda_{n-3}}, k-1, k, k)y_{\lambda_{n-3}}y_{n-2}^{a_{k-1}}y_{n-1}^{a_k}y_n^{a_k}$ . In this case we have

$$\begin{aligned} e(k-1, k, k)y_{n-2}^{a_{k-1}}y_{n-1}^{a_k}y_n^{a_k} &\stackrel{(2.6)}{=} -e(k-1, k, k)y_{n-2}^{a_{k-1}}y_{n-1}^{a_k+1}y_n^{a_k}\psi_{n-1} + e(k-1, k, k)y_{n-2}^{a_{k-1}}y_{n-1}^{a_k}y_n^{a_k}\psi_{n-1}y_n \\ &\stackrel{(2.13)}{=} \psi_{n-1}e(k-1, k, k)y_{n-2}^{a_{k-1}}y_{n-1}^{a_k}y_n^{a_k}y_n, \end{aligned}$$

where  $e(k-1, k, k)y_{n-2}^{a_{k-1}}y_{n-1}^{a_k+1}y_n^{a_k} = 0$  by induction. Therefore,

$$\begin{aligned} e(k-1, k, k)y_{n-2}^{a_{k-1}}y_{n-1}^{a_k}y_n^{a_k} &= \psi_{n-1}e(k-1, k, k)y_{n-2}^{a_{k-1}}y_{n-1}^{a_k}y_n^{a_k}y_n \\ &= \psi_{n-1}^2e(k-1, k, k)y_{n-2}^{a_{k-1}}y_{n-1}^{a_k}y_n^{a_k}y_n^2 = 0 \end{aligned}$$

by relation (2.8).

**Case 3.9b:**  $k = i_{n-1} + 1$ . Now,  $e(\mathbf{i}_\lambda \vee k)y_{\lambda_-}y_n^{b_k^\lambda} = e(\mathbf{i}_{\lambda_{n-3}}, k-2, k-1, k)y_{\lambda_{n-3}}y_{n-2}^{a_{k-2}}y_{n-1}^{a_{k-1}}y_n^{a_k+1}$ . Therefore,

$$\begin{aligned} &e(k-2, k-1, k)y_{n-2}^{a_{k-2}}y_{n-1}^{a_{k-1}}y_n^{a_k+1} \\ &\stackrel{(2.8)}{=} e(k-2, k-1, k)y_{n-2}^{a_{k-2}}y_{n-1}^{a_{k-1}+1}y_n^{a_k} + e(k-2, k-1, k)y_{n-2}^{a_{k-2}}y_{n-1}^{a_{k-1}}y_n^{a_k}\psi_{n-1}^2 \\ &\stackrel{(2.6)}{\stackrel{(2.7)}{=}} e(k-2, k-1, k)y_{n-2}^{a_{k-2}}y_{n-1}^{a_{k-1}+1}y_n^{a_k} + \psi_{n-1}e(k-2, k, k-1)y_{n-2}^{a_{k-2}}y_{n-1}^{a_k}y_n^{a_{k-1}}\psi_{n-1} = 0, \end{aligned}$$

where the last equality follows by induction.

**Case 3.9c:**  $k = i_{n-1} - 1$ . If  $n = 2$  then  $e(\mathbf{i}_\lambda \vee k)y_{\lambda_-}y_n^{b_k^\lambda} = e(k, k-1)y_1^{a_k}y_2^{a_{k-1}}$ . Then  $a_{k-1} \geq 1$ . Therefore,

$$e(k, k-1)y_1^{a_k}y_2^{a_{k-1}} \stackrel{(2.8)}{=} e(k, k-1)y_1^{a_k+1}y_2^{a_{k-1}-1} - \psi_1e(k-1, k)y_1^{a_{k-1}-1}y_2^{a_k}\psi_1 = 0,$$

using relation (2.19) and induction. Hence, the lemma follows in this case when  $n = 2$ .

If  $n > 2$  then  $e(\mathbf{i}_\lambda \vee k)y_{\lambda_-}y_n^{b_k^\lambda} = e(\mathbf{i}_{\lambda_{n-3}}, k, k+1, k)y_{\lambda_{n-3}}y_{n-2}^{a_k}y_{n-1}^{a_{k+1}}y_n^{a_k}$ . Hence,

$$\begin{aligned} e(k, k+1, k)y_{n-2}^{a_k}y_{n-1}^{a_{k+1}}y_n^{a_k} &\stackrel{(2.9)}{=} \psi_{n-2}\psi_{n-1}\psi_{n-2}e(k, k+1, k)y_{n-2}^{a_k}y_{n-1}^{a_{k+1}}y_n^{a_k} \\ &\quad - \psi_{n-1}\psi_{n-2}\psi_{n-1}e(k, k+1, k)y_{n-2}^{a_k}y_{n-1}^{a_{k+1}}y_n^{a_k} \\ &= \psi_{n-2}\psi_{n-1}e(k+1, k, k)y_{n-2}^{a_k}y_{n-1}^{a_k}y_n^{a_k}\psi_{n-2} \\ &\quad - \psi_{n-1}\psi_{n-2}e(k, k, k+1)y_{n-2}^{a_k}y_{n-1}^{a_k}y_n^{a_{k+1}}\psi_{n-1} \\ &= 0, \end{aligned}$$

where the last equality follows by induction.

**Case 3.9d:**  $|k - i_{n-1}| > 1$ . Because  $i_{n-2} = i_{n-1} - 1$ , we have  $i_{n-2} \neq k$ . Therefore we have

$$\begin{aligned} e(\mathbf{i}_{\lambda} \vee k) y_{\lambda} y_n^{b_k^{\lambda-}} &= e(\mathbf{i}_{|n-3}, i_{n-2}, i_{n-1}, k) y_{\lambda|n-3} y_{n-2}^{a_{i_{n-2}}} y_{n-1}^{a_{i_{n-1}}} y_n^{a_k} \\ &\stackrel{(2.8)}{=} \psi_{n-1} e(\mathbf{i}_{|n-3}, i_{n-2}, k, i_{n-1}) y_{\lambda|n-3} y_{n-2}^{a_{i_{n-2}}} y_{n-1}^{a_k} y_n^{a_{i_{n-1}}} \psi_{n-1} = 0 \end{aligned}$$

by induction. This completes the proof.  $\square$

Lemma 3.9 has two immediate Corollaries:

**Corollary 3.10.** Suppose  $n \geq 2$  and  $\lambda = (n|0| \dots |0)$ . Then  $e_{\lambda} y_{\lambda} y_r \in R_n^{\lambda}$  for any  $1 \leq r \leq n$ .

**Corollary 3.11.** Suppose that  $n \geq 2$  and  $\lambda = (n|0| \dots |0)$ . Then  $e_{\lambda} y_{\lambda} \psi_r \in R_n^{\lambda}$ , for any  $1 \leq r \leq n-1$ .

*Proof.* Write  $y_{\lambda} = y_1^{l_1} y_2^{l_2} \dots y_n^{l_n}$  and  $\mathbf{i}_{\lambda} = (i_1 i_2 \dots i_n)$ ,

$$e(i_1 \dots i_n) y_1^{l_1} \dots y_n^{l_n} \psi_r = \psi_r e(i_1 \dots i_{r-1} i_{r+1} i_r \dots i_n) y_1^{l_1} \dots y_{r-1}^{l_{r-1}} y_{r+1}^{l_{r+1}} y_r^{l_r} \dots y_n^{l_n} = 0,$$

by Lemma 3.9.  $\square$

The results in the rest of the subsection will be used frequently in the later proofs.

Recall that for any multipartition  $\lambda$ ,  $R_n^{\lambda}$  is the subspace of  $\mathcal{R}_n^{\Lambda}$  spanned by all of the elements  $\psi_{\text{st}}$ , where  $\text{Shape}(\mathbf{s}) = \text{Shape}(\mathbf{t}) > \lambda$ .

**Lemma 3.12.** Suppose  $\lambda \in \mathcal{S}_n^{\Lambda}$ . Then  $R_n^{\lambda}$  is a two-sided ideal of  $\mathcal{R}_n^{\Lambda}$ . More precisely,  $R_n^{\lambda}$  is a two-sided ideal of  $\mathcal{R}_n^{\Lambda}$  whenever  $\mu < \lambda$ .

*Proof.* The Lemma follows directly from the definition of the set  $\mathcal{S}_n^{\Lambda}$ ,  $\mathcal{P}_y^{\Lambda}$ ,  $\mathcal{P}_{\psi}^{\Lambda}$  and Remark 3.6.  $\square$

In order to simplify the notation, for each  $i \in I$  define  $\theta_i: \mathcal{R}_n^{\Lambda} \rightarrow \mathcal{R}_{n+1}^{\Lambda}$  to be the unique  $\mathbb{Z}$ -linear map which sends  $e(\mathbf{i})$  to  $e(\mathbf{i} \vee i)$ ,  $y_r$  to  $y_r$  and  $\psi_r$  to  $\psi_r$ . It is easy to see that  $\theta_i$  respects the relations in  $\mathcal{R}_n^{\Lambda}$ , so  $\theta_i$  is a  $\mathbb{Z}$ -algebra homomorphism.

**Lemma 3.13.** Suppose  $\lambda \in \mathcal{S}_n^{\Lambda}$  and  $\mathbf{u}, \mathbf{v} \in \text{Std}(\mu)$ , where  $\mu \in \mathcal{P}_m^{\Lambda}$  with  $m < n$  such that  $\mu > \lambda|_m$ . Let  $\sigma = \lambda|_{m+1} \in \mathcal{P}_{m+1}^{\Lambda}$ . Then  $\theta_i(\psi_{\mathbf{uv}}) \in R_n^{\sigma}$ , for any  $i \in I$ .

*Proof.* Write  $\mu = (\mu^{(1)}, \dots, \mu^{(\ell)})$  and  $\mu^{(\ell)} = (\mu_1^{(\ell)}, \dots, \mu_k^{(\ell)})$  and define  $\gamma = (\mu^{(1)}, \dots, \mu^{(\ell-1)}, \gamma^{(\ell)})$  where

$$\gamma^{(\ell)} = \begin{cases} (\mu_1^{(\ell)}, \dots, \mu_{k-1}^{(\ell)}, \mu_k^{(\ell)} + 1), & \text{if } \mu_{k-1}^{(\ell)} > \mu_k^{(\ell)}, \\ (\mu_1^{(\ell)}, \dots, \mu_{k-1}^{(\ell)}, \mu_k^{(\ell)}, 1), & \text{if } \mu_{k-1}^{(\ell)} = \mu_k^{(\ell)}. \end{cases}$$

Then  $\gamma$  is a multipartition of  $m+1$  and  $\gamma|_m = \mu$ . Since  $m < n$ , if  $m = n-1$ , then  $\gamma|_{n-1} = \mu > \lambda_-$ , so that  $\gamma > \lambda$ . On the other hand, if  $m < n-1$  then  $|\gamma| = m+1 < n = |\lambda|$ . So we always have  $\gamma < \lambda$ . Therefore,  $\gamma \in \mathcal{P}_I^{\Lambda} \cap \mathcal{P}_y^{\Lambda} \cap \mathcal{P}_{\psi}^{\Lambda}$  because  $\lambda \in \mathcal{S}_n^{\Lambda}$ .

As  $\gamma|_m = \mu$ , we have  $\theta_i(\psi_{\mathbf{uv}}) = \theta_i(\psi_{d(\mathbf{u})}^* e_{\mu} y_{\mu} \psi_{d(\mathbf{v})}) = \psi_{d(\mathbf{u})}^* e(\mathbf{i}_{\gamma|_m} \vee i) y_{\mu} \psi_{d(\mathbf{v})}$ . First suppose that  $b_i^{\mu} = 0$ . Then using the definition of  $\mathcal{P}_I^{\Lambda}$ , we have  $e(\mathbf{i}_{\gamma|_m} \vee i) y_{\mu} \in R_n^{\gamma} \subseteq R_n^{\sigma}$ . Hence, by Lemma 3.12, we have  $\theta_i(\psi_{\mathbf{uv}}) = \psi_{d(\mathbf{u})}^* e(\mathbf{i}_{\gamma|_m} \vee i) y_{\mu} \psi_{d(\mathbf{v})} \in R_n^{\sigma}$ .

Now suppose that  $b_i^{\mu} > 0$ . By Lemma 3.3 there exists a multipartition  $\nu$  with  $\nu|_m = \mu$  such that  $e(\mathbf{i}_{\gamma|_m} \vee i) y_{\mu} = e_{\nu} y_{\nu}$ . Further, as  $\nu|_m = \mu$ , there exist two standard  $\nu$ -tableaux  $\mathbf{s}$  and  $\mathbf{t}$  such that  $\mathbf{s}|_m = \mathbf{u}$  and  $\mathbf{t}|_m = \mathbf{v}$ . That is,  $d(\mathbf{s}) = d(\mathbf{u})$  and  $d(\mathbf{t}) = d(\mathbf{v})$ . Therefore,

$$\theta_i(\psi_{\mathbf{uv}}) = \psi_{d(\mathbf{u})}^* e(\mathbf{i}_{\gamma|_m} \vee i) y_{\mu} \psi_{d(\mathbf{v})} = \psi_{d(\mathbf{s})}^* e_{\nu} y_{\nu} \psi_{d(\mathbf{t})} = \psi_{\mathbf{st}} \in R_n^{\nu} \subseteq R_n^{\sigma}$$

because  $\nu > \sigma$ . This completes the proof.  $\square$

If  $\mathbf{i} = (i_1, \dots, i_n) \in I^n$  and  $1 \leq m \leq n$  let  $\mathbf{i}_m = (i_1 \dots i_m)$ .

**Lemma 3.14.** Suppose  $\lambda \in \mathcal{S}_n^{\Lambda}$ ,  $m \leq n$  and  $\sigma = \lambda|_m$ . For any  $\mathbf{i} = (i_1, i_2, \dots, i_{n-m})$  we have  $\mathcal{R}_n^{\Lambda} \theta_i(R_n^{\sigma}) \mathcal{R}_n^{\Lambda} \subseteq R_n^{\lambda}$ .

*Proof.* Suppose  $r \in R_n^{\sigma}$ , we have that

$$r = \sum_{\mathbf{u}, \mathbf{v} \in \text{Std}(>\sigma)} c_{\mathbf{uv}} \psi_{\mathbf{uv}}$$

for some  $c_{\mathbf{uv}} \in \mathbb{Z}$ . For any  $i \in I$ ,

$$\theta_i(r) = \sum_{\mathbf{u}, \mathbf{v} \in \text{Std}(>\sigma)} c_{\mathbf{uv}} \theta_i(\psi_{\mathbf{uv}}).$$

By Lemma 3.13,  $\theta_i(\psi_{\mathbf{uv}}) \in R_n^{\lambda|_{m+1}}$ . Hence  $\theta_i(r) \in R_n^{\lambda|_{m+1}}$ . By induction we have  $\theta_i(r) \in R_n^{\lambda}$ . By Lemma 3.12,  $R_n^{\lambda}$  is an ideal. Therefore  $\mathcal{R}_n^{\Lambda} \theta_i(R_n^{\sigma}) \mathcal{R}_n^{\Lambda} \subseteq R_n^{\lambda}$  which completes the proof.  $\square$

### 3.2. The action of $y_r$ on two-rowed partitions

Recall that the main result of this section is to prove that if  $\lambda \in \mathcal{S}_n$ , then

$$e(\mathbf{i}_{\lambda} \vee k)y_{\lambda}y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

In the inductive process we consider different types of multipartitions  $\lambda$  and a residue  $k \in I$ . We will consider the more difficult case first, namely when  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  and  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_l^{(\ell)}, 1) \neq \emptyset$  with  $l \geq 2$ ,  $\lambda_{l-1}^{(\ell)} = \lambda_l^{(\ell)} = m$  and  $k \equiv \kappa_\ell - l + m + 1 \pmod{e}$ . In this subsection we assume that  $\ell = 1$  and  $l = 2$ . We will extend the result to the general case in the next subsection. Notice that in this case  $\lambda = (m, m, 1)$  for some integer  $m$  and  $k \equiv \kappa_1 - 1 + m \pmod{e}$ . Then  $e(\mathbf{i}_{\lambda} \vee k)y_{\lambda}y_n^{b_k^{\lambda_-}} = e_{\gamma}y_{\gamma}$  where  $\gamma = (m, m+1)$ . It is very hard to prove that  $e_{\gamma}y_{\gamma} \in R_n^{>\lambda}$  directly, so we are going to work with  $\gamma$  which is in a more general form.

In this subsection we fix  $\Lambda = \Lambda_j$  for some  $j \in I$ ,  $\gamma = (\gamma_1, \gamma_2)$  and  $\lambda = (\gamma_1, \gamma_2 - 1, 1)$  with  $\gamma_2 > 1$  and  $\gamma_2 - \gamma_1 \equiv 1 \pmod{e}$ . We will prove that if  $\gamma_1 + 1 = \gamma_2$  and  $\lambda \in \mathcal{S}_n^{\Lambda}$  then  $e_{\gamma}y_{\gamma} \in R_n^{>\gamma}$ .

Without loss of generality we can assume that  $\Lambda = \Lambda_0$ . Define  $i \equiv \gamma_2 - 2 \pmod{e}$ , which is the residue of  $(2, \gamma_2, 1)$ . Because  $\gamma_2 \equiv \gamma_1 + 1 \pmod{e}$ , it is also the residue of the node  $(1, \gamma_1, 1)$ . In diagrammatic notation, we have

$$e_{\gamma}y_{\gamma} = \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & & i-1 & i & e-1 & 0 & i-1 & i \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bullet_{i_1} & \bullet_{i_2} & & \bullet_{i_m-1} & \bullet_{i_m} & \bullet_{i_{m+1}} & \bullet_{i_{m+2}} & \bullet_{i_{2m}} & \bullet_{i_{2m+1}} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline & & & \gamma_1 & & \gamma_2 & & & \end{array} \end{array}$$

where  $\mathbf{i}_{\gamma} = (i_1, i_2, \dots, i_n)$  and  $l_k = |\mathcal{A}_{\mathbf{v}|_k}^{i_k}|$  is the multiplicity of the green dot on the  $k$ -th string. For the rest of this subsection, for clarity we will omit extraneous dots when they do not play an important role in the argument.

Next we introduce an important equivalent relation  $=_{\gamma}$ . For  $\gamma \in \mathcal{S}_n^{\Lambda}$ , and  $r_1, r_2 \in \mathcal{R}_n^{\Lambda}$ , we write  $r_1 =_{\gamma} r_2$  if  $r_1 \pm r_2 \in R_n^{>\gamma}$ . It is clearly an equivalent relation. Moreover, by Lemma 3.12, for any  $r \in \mathcal{R}_n^{\Lambda}$  we have  $r_1 \cdot r =_{\gamma} r_2 \cdot r$  if  $r_1 =_{\gamma} r_2$ . This will be helpful for us to simplify the notations and calculations.

Recall that  $\gamma_2 > 1$ . We can write  $\gamma_2 = k \cdot e + t$  for some nonnegative integer  $k$  and  $2 \leq t \leq e + 1$ . We will first prove

$$e_{\gamma}y_{\gamma} =_{\gamma} \begin{cases} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & & i-1 & i & e-1 & 0 & i-1 & i \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline & & & \gamma_1 & & \gamma_2 & & & \end{array} \end{array}, & \text{if } i \neq e-1, \\ \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & & e-2e-1e-1 & 0 & e-2e-1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline & & & \gamma_1 & & \gamma_2 & & & \end{array} \end{array}, & \text{if } i = e-1. \end{cases} \quad (3.15)$$

by induction on  $k$ , which can imply  $e_{\gamma}y_{\gamma} \in R_n^{>\lambda}$  easily.

In order to clarify the meaning of the diagrams in (3.15), let us give two examples below. In these examples for convenience we fix  $e = 4$ .

**Example 3.16.** Suppose  $\gamma = (8, 5)$ , then  $\gamma = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$  and  $i = 3$ . Then we are trying to prove that

$$e_{\gamma}y_{\gamma} = \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bullet & & & & \bullet & & & & \bullet & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline & & & \gamma_1 & & \gamma_2 & & & & & & \end{array} \end{array} =_{\gamma} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 3 & 0 & 1 & 2 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline & & & \gamma_1 & & \gamma_2 & & & & & & \end{array} \end{array}.$$

**Example 3.17.** Suppose  $\gamma = (9, 10)$ , then  $\gamma = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline & & & & & & & & & \\ \hline \end{array}$  and  $i = 0$ . We are trying to prove that

$$\begin{aligned} e_\gamma y_\gamma &= \begin{array}{c} \begin{array}{cccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} \\ &=_{\gamma} \begin{array}{c} \begin{array}{cccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} \end{aligned}$$

The next Proposition is the base case of the induction. When  $k = 0$ , we have  $2 \leq \gamma_2 \leq e + 1$ .

**Proposition 3.18.** Suppose  $\gamma = (\gamma_1, \gamma_2) \in \mathcal{C}_n^\Lambda$  with  $\gamma_2 > 1$  and  $\gamma_2 - \gamma_1 \equiv 1 \pmod{e}$  and  $\lambda = (\gamma_1, \gamma_2 - 1, 1) \in \mathcal{S}_n^\Lambda$ . Define  $i$  to be the residue of the node at position  $(1, \gamma_1, 1)$  or  $(2, \gamma_2, 1)$ . When  $2 \leq \gamma_2 \leq e + 1$ , (3.15) holds.

Before proving Proposition 3.18 we first give a useful lemma.

**Lemma 3.19.** For any  $i \in I$ , we have

$$\begin{array}{c} \begin{array}{cccccccc} i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} = \begin{array}{c} \begin{array}{cccccccc} i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} - \begin{array}{c} \begin{array}{cccccccc} i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} - \begin{array}{c} \begin{array}{cccccccc} i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} + \begin{array}{c} \begin{array}{cccccccc} i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} + \begin{array}{c} \begin{array}{cccccccc} i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i & i & i+1i+2 & i-1 & i \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} \end{array}$$

*Proof.* The Lemma follows by directly applying braid relations on the left hand side of the equation.  $\square$

Now we are ready to prove Proposition 3.18.

*Proof.* We prove the Proposition by considering four different cases depending upon the value of  $i$ . Notice that in this Proposition, we have  $\gamma_1 \geq \gamma_2 - 1$  because  $2 \leq \gamma_2 \leq e + 1$  and  $\gamma_2 - \gamma_1 \equiv 1 \pmod{e}$ .

**Case 3.18a:**  $i = 0$ , i.e.  $\gamma_2 = 2$ .

$$\begin{aligned} e_\gamma y_\gamma &= \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} \stackrel{(2.12)}{=} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} - \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} \\ &\stackrel{(2.17)}{=} - \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} - \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} = - \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} - \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} \end{aligned}$$

Because

$$\begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} = \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} - \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 & 0 & 1 & e-1 & 0 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} \end{array}$$

and if we define  $\nu = (\gamma_1, 1) = \lambda|_{\gamma_1+1}$ , then as  $\lambda \in \mathcal{S}_n^\Lambda$  and  $|\nu| = \gamma_1 + 1 < n = \gamma_1 + \gamma_2 = |\lambda|$ ,  $\nu \in \mathcal{P}_I^\Lambda$ . Moreover as  $b_0^{\nu_-} = 1$ ,

$$\begin{array}{c} \begin{array}{cccc} 0 & 1 & e-1 & 0 \end{array} \\ \begin{array}{|c|c|c|c|} \hline \cdot \\ \hline \end{array} \end{array} = e(\mathbf{i}_{\nu_-} \vee 0) y_{\nu_-} y_{|\nu|}^1 \in R_n^{>\nu}.$$



Then by Lemma 3.14,

$$\in R_n^{>\gamma}.$$

Therefore,

$$e_\gamma y_\gamma =_\gamma,$$

which gives the proposition in this case.

**Case 3.18b:**  $1 \leq i \leq e-3$ , i.e.  $3 \leq \gamma_2 \leq e-1$ .

$$\begin{aligned} & \text{(2.12)} \\ & \text{(2.12)} \\ & \text{(2.11)} \\ & = \end{aligned}$$

For the same reason as in Case 3.18a,

$$\in R_n^{>\gamma},$$

which implies the proposition in this case.

**Case 3.18c:**  $i = e-2$ , i.e.  $\gamma_2 = e$ . By Lemma 3.19,

$$\begin{aligned} & = \\ & - \\ & + \end{aligned}$$

Set  $\nu = (\gamma_1, \gamma_2 - 1) = \gamma|_{n-1}$ . As  $\gamma_1 \geq \gamma_2 - 1$ , we have  $\nu \in \mathcal{P}_{n-1}^\Lambda$ . As  $\lambda \in \mathcal{S}_n^\Lambda$  and  $|\nu| < |\lambda|$ , we have  $\nu \in \mathcal{P}_l^\Lambda$ . It is not hard to see that  $b_{e-3}^{\nu_-} = 1$ . Hence

$$\begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad 0 \quad e-4e-3 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} = e(\mathbf{i}_{\nu_-} \vee e-3)\gamma_{\nu_-}\gamma_{n-1}^1 \in R_n^{>\nu}.$$

$\underbrace{\hspace{10em}}_{\gamma_1} \quad \underbrace{\hspace{10em}}_{\gamma_2-1}$

Then by Lemma 3.14,

$$\begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad 0 \quad e-3e-2 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} \in R_n^{>\gamma}.$$

Similarly, we have

$$\begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad 0 \quad e-3e-2 \quad 0 \quad 1 \quad e-3e-2e-1 \quad 0 \quad e-3e-2 \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array}, \quad \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad 0 \quad e-3e-2 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} \in R_n^{>\gamma},$$

and for the similar argument as in Case 3.18a, we have

$$\begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad 0 \quad e-3e-2 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} \in R_n^{>\gamma}.$$

Therefore,

$$\begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad 0 \quad e-3e-2 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} \xrightarrow{\gamma} \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad 0 \quad e-3e-2 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} = e_\gamma \gamma_\gamma,$$

which follows the Proposition.

**Case 3.18d:**  $i = e - 1$ , i.e.  $\gamma_2 = e + 1$ . By Lemma 3.19,

$$\begin{array}{c} \begin{array}{c} 0 \quad 1 \quad e-2e-1e-1 \quad 0 \quad 1 \quad e-3e-2e-1 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} = \begin{array}{c} 0 \quad 1 \quad e-2e-1e-1 \quad 0 \quad 1 \quad e-3e-2e-1 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} \\ - \begin{array}{c} 0 \quad 1 \quad e-2e-1e-1 \quad 0 \quad 1 \quad e-3e-2e-1 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} - \begin{array}{c} 0 \quad 1 \quad e-2e-1e-1 \quad 0 \quad 1 \quad e-3e-2e-1 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} \\ + \begin{array}{c} 0 \quad 1 \quad e-2e-1e-1 \quad 0 \quad 1 \quad e-3e-2e-1 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} + \begin{array}{c} 0 \quad 1 \quad e-2e-1e-1 \quad 0 \quad 1 \quad e-3e-2e-1 \\ \begin{array}{|c|c|c|c|c|c|} \hline \vdots \\ \hline \end{array} \end{array} \end{array} \quad (3.20)$$

Figure 1 consists of two rows of diagrams, labeled (2.11) and (2.15). Each diagram is a sequence of vertical blue lines. Above each line is a label: 0 or 1. The labels are grouped into four sets of three lines each. In (2.11), the first set has labels 0, 1,  $e-2e-1e-1$ ; the second has 0, 1,  $e-3e-2e-1$ ; the third has 0, 1,  $e-2e-1e-1$ ; and the fourth has 0, 1,  $e-3e-2e-1$ . The diagrams show various configurations of green dots and crossings. In (2.15), the diagrams are similar but include red dashed boxes around certain crossings and dots. The first diagram in (2.15) has a red dashed box around a crossing of two lines, with a green dot on each line. The second diagram has a red dashed box around a crossing of two lines, with a green dot on one line. The third diagram has a red dashed box around a crossing of two lines, with a green dot on one line. The fourth diagram has a red dashed box around a crossing of two lines, with a green dot on one line.

$$(2.15) \quad - \left( \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \right) \in R_n^{\geq \gamma}.$$

The figure shows a sequence of seven diagrams illustrating the proof of Lemma 10. The diagrams are arranged in three rows. The first row shows the initial expression as a difference of two terms. The second row shows the first term of the first row being further decomposed into a difference of two terms. The third row shows the final result as a difference of two terms, which are identified as  $e_\gamma y_\gamma$  and  $e_\gamma y_\gamma$  respectively. Each diagram consists of vertical blue lines with green dots and red dashed boxes. Labels above the lines are binary strings like "0 1  $e-2e-1e-1$  0 1  $e-3e-2e-1$ ".

and for the last term of (3.20),

$$\in R_n^{>\gamma}.$$

Combine the result above, we have

$$=_{\gamma} e_{\gamma} y_{\gamma},$$

which completes the proof. □

**Remark 3.21.** The technique of applying Lemma 3.14 in proving Proposition 3.18 will be used many times in the rest of the paper. Although the process is straightforward, individual details will vary from case to case, thus in order to simplify the process we will omit details in the future.

Recall  $\gamma_2 = k \cdot e + t$  where  $k$  is a nonnegative integer and  $2 \leq t \leq e + 1$ . Now we remove the restriction on  $\gamma_2$  by applying the induction on  $k$ .

**Proposition 3.22.** Suppose  $\gamma = (\gamma_1, \gamma_2) \in \mathcal{C}_n^{\Lambda}$  with  $\gamma_2 > 1$  and  $\gamma_2 - \gamma_1 \equiv 1 \pmod{e}$  and  $\lambda = (\gamma_1, \gamma_2 - 1, 1) \in \mathcal{S}_n^{\Lambda}$ . Define  $i$  to be the residue of the node at position  $(1, m, 1)$ . Then (3.15) holds.

*Proof.* We prove this Proposition by induction. As we claimed before that we can write  $\gamma_2 = k \cdot e + t$  with  $2 \leq t \leq e + 1$  and we will apply induction on  $k$ . Proposition 3.18 implies that for  $k = 0$  the Proposition holds. Assume that for  $k \leq k'$  the Proposition holds. For  $k = k'$ , we consider two different cases, which are  $i = e - 2, i = e - 1$  and  $i \neq e - 2, e - 1$ . Recall that  $i$  is the residue of the node at  $(1, m, 1)$  or  $(2, m + 1, 1)$ .

**Case 3.22a:**  $i \neq e - 2, e - 1$ .

$$\begin{aligned}
 & \begin{array}{c} \text{Diagram 1} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} \stackrel{(2.10)}{=} \begin{array}{c} \text{Diagram 2} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} \\
 & = \begin{array}{c} \text{Diagram 3} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} \quad \text{by Lemma 3.19} \\
 & = \begin{array}{c} \text{Diagram 4} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} - \begin{array}{c} \text{Diagram 5} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} \\
 & - \begin{array}{c} \text{Diagram 6} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} + \begin{array}{c} \text{Diagram 7} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} \\
 & + \begin{array}{c} \text{Diagram 8} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} \\
 & \stackrel{(2.11)}{=} \begin{array}{c} \text{Diagram 9} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} + \begin{array}{c} \text{Diagram 10} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} \\
 & \stackrel{(2.17)}{=} \begin{array}{c} \text{Diagram 11} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} + \begin{array}{c} \text{Diagram 12} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} \\
 & \stackrel{(2.10)}{=} \begin{array}{c} \text{Diagram 13} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} + \begin{array}{c} \text{Diagram 14} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} .
 \end{aligned}$$

Then by induction and Lemma 3.14, we have

$$\begin{array}{c} \text{Diagram 15} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} =_{\gamma} \begin{array}{c} \text{Diagram 16} \\ \gamma_1 \quad \gamma_2^{-e} \quad e \end{array} = 0,$$

which implies that

$$\in R_n^{>\gamma}.$$

Hence by induction and Lemma 3.14

$$\equiv_{\gamma} \ell_{\gamma} y_{\gamma}.$$

**Case 3.22b:**  $i = e - 2$ .

by Lemma 3.19

(3.23)

By induction and Lemma 3.14, the second and the third terms of (3.23) are both in  $R_n^{>\gamma}$ . Now for the last term.

$$\begin{aligned}
 & \begin{array}{c} \text{Diagram 1} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \\
 & \stackrel{(2.10)}{=} \begin{array}{c} \text{Diagram 2} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} - \begin{array}{c} \text{Diagram 3} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array}. \tag{3.24}
 \end{aligned}$$

Substitute (3.24) to (3.23), let  $n = \gamma_1 + \gamma_2$ , we have

$$\begin{aligned}
 & \begin{array}{c} \text{Diagram 4} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} = \begin{array}{c} \text{Diagram 5} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \\
 & + \begin{array}{c} \text{Diagram 6} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} + \begin{array}{c} \text{Diagram 7} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \\
 & = (y_n + \begin{array}{c} \text{Diagram 8} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array}) \cdot \begin{array}{c} \text{Diagram 9} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \\
 & + \begin{array}{c} \text{Diagram 10} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array}, \tag{3.25}
 \end{aligned}$$

where by Lemma 3.19

$$\begin{aligned}
 & y_n + \begin{array}{c} \text{Diagram 11} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \\
 & = \begin{array}{c} \text{Diagram 12} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} + \begin{array}{c} \text{Diagram 13} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array}
 \end{aligned}$$

$$\begin{aligned}
& - \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) + \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \\
& + \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) - \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \\
& = \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) + \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \\
& + \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) - \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right).
\end{aligned} \tag{3.26}$$

Then by  $\lambda \in \mathcal{S}_n^\Lambda$  and Lemma 3.14, for the first term of (3.25),

$$\begin{aligned}
& (y_n + \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \cdot \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \text{ by induction} \\
& =_{\gamma} (y_n + \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \cdot \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \text{ by (3.26)} \\
& = \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) + \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \text{ by Lemma 3.14} \\
& + \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) - \left( \begin{array}{c} 0 \quad 1 \quad e-3e-2e-1 \quad e-3e-2e-1 \quad e-3e-2 \\ \hline \gamma_1 \quad \gamma_2-e \quad e \end{array} \right) \in R_n^{>\gamma}. \tag{3.27}
\end{aligned}$$



For the second term of (3.25), by induction, Lemma 3.19,  $\lambda \in \mathcal{S}_n^\Lambda$  and Lemma 3.14,

$$\begin{aligned}
 & \begin{array}{c} \text{Diagram 1} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \stackrel{(2.11)}{=} \begin{array}{c} \text{Diagram 2} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \quad \text{by induction} \\
 & \stackrel{=_{\gamma}}{=} \begin{array}{c} \text{Diagram 3} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \stackrel{(2.10)}{=} \begin{array}{c} \text{Diagram 4} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \quad \text{by Lemma 3.19} \\
 & \stackrel{=}{=} \begin{array}{c} \text{Diagram 5} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} - \begin{array}{c} \text{Diagram 6} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \\
 & - \begin{array}{c} \text{Diagram 7} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} + \begin{array}{c} \text{Diagram 8} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \\
 & \stackrel{=}{=} \begin{array}{c} \text{Diagram 9} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \stackrel{\text{by Lemma 3.14}}{=} \begin{array}{c} \text{Diagram 10} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \\
 & \stackrel{=_{\gamma}}{=} \begin{array}{c} \text{Diagram 11} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} = e_{\gamma} y_{\gamma}. \tag{3.28}
 \end{aligned}$$

Substitute the results of (3.27) and (3.28) to (3.25), we have

$$\begin{array}{c} \text{Diagram 12} \\ \gamma_1 \quad \gamma_2 - e \quad e \end{array} \stackrel{=_{\gamma}}{=} e_{\gamma} y_{\gamma}.$$

**Case 3.22c:**  $i = e - 1$ . The method to prove this is the same as for Case 3.22a so it is left as an exercise. Then by induction, this completes the proof.  $\square$

Finally, we can use (3.15) to prove our main result of this subsection.

**Proposition 3.29.** Suppose  $m$  is a positive integer,  $\lambda = (m, m, 1) \in \mathcal{S}_n^\Lambda$  and  $\gamma = (\gamma_1, \gamma_2) = (m, m + 1)$ . Recall  $\lambda_- = (m, m)$ . Write  $\mathbf{i}_{\lambda_-} = (i_1, i_2, \dots, i_{n-1})$ . If  $k = i_{n-1} + 1 \in I$ , we have

$$e_{\gamma} y_{\gamma} = e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} b_k^{\lambda_-} \in R_n^{\lambda}.$$

*Proof.* Without loss of generally we assume  $\Lambda = \Lambda_0$ . When  $m = 1$ , then  $\gamma = (1, 2)$  and

$$e_\gamma y_\gamma = \begin{array}{|c|c|c|} \hline 0 & e-1 & 0 \\ \hline \hline \hline \end{array} \stackrel{(2.12)}{=} \stackrel{(2.17)}{=} \begin{array}{|c|c|} \hline 0 & e-1 & 0 \\ \hline \hline \hline \end{array} - \begin{array}{|c|c|} \hline 0 & e-1 & 0 \\ \hline \hline \hline \end{array} \in N_3^{\Lambda_0} \subseteq R_n^{\geq \lambda}.$$

When  $m > 1$ , write  $\mathbf{i}_{(\gamma_1)} = (i_1, i_2, \dots, i_m)$ . Set  $\sigma = (\gamma_1 - 1, \gamma_2) = (m-1, m)$ . Then by Proposition 3.22, we have

$$e_\gamma y_\gamma = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & i_m-1 & i_m & e-1 & 0 & i_m-1 & i_m \\ \hline \hline \hline \hline \hline \hline \hline \end{array} =_\gamma \begin{cases} \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & i_m-1 & i_m & e-1 & 0 & i_m-1 & i_m \\ \hline \hline \hline \hline \hline \hline \hline \end{array}, & \text{if } i_m \neq e-1, \\ \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & e-2e-1 & e-1 & 0 & e-2e-1 \\ \hline \hline \hline \hline \hline \hline \hline \end{array}, & \text{if } i_m = e-1. \end{cases}$$

In both cases, the parts bounded by square are both  $e_\sigma y_\sigma$ . As  $|\sigma| = n-2$  and  $\lambda \in \mathcal{S}_n^\Lambda$ , by induction,  $e_\sigma y_\sigma \in R_n^{\geq \sigma}$ . By the definition of  $\sigma$  and Lemma 2.36, it forces that  $e_\sigma y_\sigma \in R_n^{\geq \lambda_{n-2}}$ . Then by Lemma 3.14, we have  $e_\gamma y_\gamma \in R_n^{\geq \lambda}$ .  $\square$

### 3.3. Final part of y-problem

In the last subsection we have proved that if  $\lambda = (m, m, 1) \in \mathcal{S}_n^\Lambda$ , then

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{\geq \lambda}$$

with  $k = i_n + 1$ . In this subsection we will gradually remove the restrictions on  $\lambda$  and  $k$ . First we are going to introduce a useful homomorphism and use it to prove a few more properties of  $\mathcal{R}_n$  and  $\mathcal{R}_n^\Lambda$ . After that we are going to show that if  $\lambda \in \mathcal{P}_I^\Lambda$ , then we can extend  $\lambda$  to a  $\ell + 1$  multipartition by adding an  $\emptyset$  at the end and thus the new multipartition is in  $\mathcal{P}_I^{\Lambda + \Lambda_i}$  for any  $i \in I$ . Analogous results are also true for  $\mathcal{P}_y^\Lambda$  and  $\mathcal{P}_\psi^\Lambda$ . These will allow us to extend the result to an arbitrary multipartition  $\lambda$ .

For any  $\mathbf{j} \in I^m$ , we can define a linear map  $\hat{\theta}_{\mathbf{j}} : \mathcal{R}_n \rightarrow \mathcal{R}_{n+m}^\Lambda$  sending  $e(\mathbf{i})$  to  $e(\mathbf{j} \vee \mathbf{i})$ ,  $y_r$  to  $y_{r+m}$  and  $\psi_r$  to  $\psi_{r+m}$ . This map  $\hat{\theta}_{\mathbf{j}}$  works as embedding from  $\mathcal{R}_n$  to  $\mathcal{R}_{n+m}$  followed by the projection onto  $\mathcal{R}_{n+m}^\Lambda$ .

**Lemma 3.30.** *For  $\mathbf{j} \in I^m$ , the map  $\hat{\theta}_{\mathbf{j}}$  is a homomorphism.*

*Proof.* The map is defined to be linear. Hence we only have to check the relations. Since the relations of  $\mathcal{R}_n$  and  $\mathcal{R}_{n+m}^\Lambda$  from Definition 2.1 are independent of the value of  $r$ , we see that  $\hat{\theta}_{\mathbf{j}}$  is a homomorphism.  $\square$

It will be necessary to cut a multicomposition  $\lambda$  into one multicomposition  $\mu$  and a composition  $\gamma$  for our later work. Note that in our work we will mainly set  $\mu$  to be a multipartition and  $\gamma$  to be a partition, but generally we don't have such restriction.

**Example 3.31.** Fix  $e = 4$ ,  $\Lambda = 2\Lambda_0 + \Lambda_1$ ,  $\kappa_\Lambda = (0, 1, 0)$ . Suppose  $\lambda = (4, 2|2^2, 1|3^2, 2)$ . So

$$[\lambda] = \left( \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \middle| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \middle| \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right).$$

We want to divide the last partition of  $\lambda$  after the first row. This is called the cut row of  $\lambda$ . This gives us a multipartition  $\mu$  with Young diagram

$$[\mu] = \left( \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \middle| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \middle| \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right),$$

and a partition  $\gamma$  with diagram

$$[\gamma] = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}.$$

We call  $\mu$  and  $\gamma$  the cut part and the remaining part, respectively.

Moreover we want to preserve the following data. The value  $|\mu|$  is called the cut of  $\lambda$  which is 14 in this case. The residue of the top left node of  $\gamma$  as a subdiagram of  $\lambda^{(3)}$  is called the cut residue, which in this case is 3.

Now we give a formal definition.

**Definition 3.32.** Suppose  $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)}) \in \mathcal{C}_n^\Lambda$  with  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_{k_\ell}^{(\ell)})$  and  $a$  is an integer such that  $0 \leq a < k_\ell$ . We call  $m$  a **cut** of  $\lambda$  and  $a$  the **cut row** associated to  $m$  where  $m = \sum_{i=1}^{\ell-1} |\lambda^{(i)}| + \sum_{j=1}^a \lambda_j^{(\ell)}$ . Define  $\Lambda' = \Lambda_s$ , where  $s = k_\ell + 1 - (a+1) = k_\ell - a$ , the residue of the node at position  $(a+1, 1, \ell)$ . We call  $s$  to be **cut residue** associated to  $m$  and  $\Lambda'$  to be **cut weight** associated to  $m$ . We then define  $\mu = \lambda|_m \in \mathcal{C}_m^\Lambda$  and  $\gamma = (\lambda_{a+1}^{(\ell)}, \lambda_{a+2}^{(\ell)}, \dots, \lambda_{k_\ell}^{(\ell)}) \in \mathcal{C}_{n-m}^{\Lambda'}$  and call  $\mu$  and  $\gamma$  to be **cut part** and **remaining part** of  $\lambda$  associated to  $m$ , respectively.

Note we can either remove a portion of the last tableau, or cut out the whole partition.

We will start to work with  $\hat{\theta}_i$ , which involving elements in both  $\mathcal{R}_n$  and  $\mathcal{R}_n^\Lambda$ . Recall that  $\hat{e}(\mathbf{i})$ ,  $\hat{y}_r$ ,  $\hat{\psi}_s$  and  $\hat{\psi}_{st}$  are elements from  $\mathcal{R}_n$  and  $e(\mathbf{i})$ ,  $y_r$ ,  $\psi_s$  and  $\psi_{st}$  are elements from  $\mathcal{R}_n^\Lambda$ .

**Lemma 3.33.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$ . Let  $m$  be a cut of  $\lambda$  with  $m < n - 1$ ,  $\nu = \lambda|_m$  and  $\Lambda'$  be the cut weight associated to  $m$ . Consider  $N_{n-m}^{\Lambda'} \subseteq \mathcal{R}_{n-m}$ . If  $\hat{\theta}_{i_\nu} : \mathcal{R}_{n-m} \rightarrow \mathcal{R}_n^\Lambda$ , then  $\hat{\theta}_{i_\nu}(N_{n-m}^{\Lambda'})_{y_\nu} \subseteq R_n^{\geq \lambda}$ .

*Proof.* Consider  $r \in N_{n-m}^{\Lambda'}$ . Then by (2.40),

$$r = \sum_{\mathbf{j}=(j_1, j_2, \dots, j_{n-m}) \in I^{n-m}} c_j R'_j \hat{e}(\mathbf{j}) \hat{y}_1^{(\Lambda', \alpha_{j_1})} R_j,$$

where  $R_j$  and  $R'_j$  are some elements in  $\mathcal{R}_{n-m}$  and  $c_j \in \mathbb{Z}$ . Therefore

$$\begin{aligned} \hat{\theta}_{i_\nu}(r)_{y_\nu} &= \sum_{\mathbf{j}=(j_1, j_2, \dots, j_{n-m}) \in I^{n-m}} c_j \hat{\theta}_{i_\nu}(R'_j) \hat{\theta}_{i_\nu}(\hat{e}(\mathbf{j}) \hat{y}_1^{(\Lambda', \alpha_{j_1})}) \hat{\theta}_{i_\nu}(R_j)_{y_\nu} \\ &= \sum_{\mathbf{j}=(j_1, j_2, \dots, j_{n-m}) \in I^{n-m}} c_j \hat{\theta}_{i_\nu}(R'_j) e(\mathbf{i}_\nu \vee j_1 \vee j_2 j_3 \dots j_{n-m})_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}} \hat{\theta}_{i_\nu}(R_j) \\ &= \sum_{\mathbf{j}=(j_1, j_2, \dots, j_{n-m}) \in I^{n-m}} c_j \hat{\theta}_{i_\nu}(R'_j) \theta_{(j_2, j_3, \dots, j_{n-m})}(e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}}) \hat{\theta}_{i_\nu}(R_j). \end{aligned}$$

Next we consider  $e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}} \in \mathcal{R}_{m+1}^\Lambda$ .

Recall that we can write  $\nu = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell-1)}, \nu^{(\ell)})$  and  $\nu^{(\ell)} = (\nu_1^{(\ell)}, \dots, \nu_l^{(\ell)})$ . Let  $\mu = \lambda|_{m+1}$ . As  $m < n - 1$ ,  $|\mu| = m + 1 < n = |\lambda|$ , and  $\lambda \in \mathcal{S}_n^\Lambda$ , we have  $\mu \in \mathcal{P}_I^\Lambda$ . Write  $\mathbf{i}_\nu = (i_1, i_2, \dots, i_m)$ . Notice that  $(\Lambda', \alpha_{j_1}) = |\mathcal{A}_\nu^{j_1}|$ . We consider two cases.

Suppose  $j_1 = i_m + 1$  and  $\nu^+$  is a multipartition. By Definition 3.2 we have  $|\mathcal{A}_\nu^{j_1}| = b_{j_1}^\nu - 1$ . Then by Lemma 3.3 we have  $e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}} = e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{b_{j_1}^\nu - 1}} = e_{\nu^+ y_{\nu^+}}$ . Because  $m$  is a cut of  $\lambda$  and  $\nu = \lambda|_m$ ,  $\mu = \lambda|_{m+1}$ , we must have  $\nu^+ > \mu$ . Therefore  $e_{\nu^+ y_{\nu^+}} \in R_n^{\geq \mu}$ . So

$$e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}} \in R_n^{\geq \mu}.$$

Otherwise, by Definition 3.2 we have  $|\mathcal{A}_\nu^{j_1}| = b_{j_1}^\nu$ . Then by  $\mu \in \mathcal{P}_I^\Lambda$  and the definition of  $\mathcal{P}_I^\Lambda$ , for any  $j_1 \in I$ , we have  $e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{b_{j_1}^\nu}} \in R_n^{\geq \mu}$  because  $\nu = \mu_-$ . Therefore

$$e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}} = e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{b_{j_1}^\nu}} \in R_n^{\geq \mu}.$$

Therefore for any  $j_1 \in I$  we have  $e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}} \in R_n^{\geq \mu}$ . Hence by Lemma 3.14 and Lemma 3.12,

$$\hat{\theta}_{i_\nu}(R'_j) \theta_{(j_2, j_3, \dots, j_{n-m})}(e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}}) \hat{\theta}_{i_\nu}(R_j) \subseteq R_n^{\geq \lambda}.$$

Therefore  $\hat{\theta}_{i_\nu}(r)_{y_\nu} = \sum_{\mathbf{j}=(j_1, j_2, \dots, j_{n-m}) \in I^{n-m}} c_j \hat{\theta}_{i_\nu}(R'_j) \theta_{(j_2, j_3, \dots, j_{n-m})}(e(\mathbf{i}_\nu \vee j_1)_{y_\nu y_{m+1}^{(\Lambda', \alpha_{j_1})}}) \hat{\theta}_{i_\nu}(R_j) \subseteq R_n^{\geq \lambda}$ .  $\square$

**Definition 3.34.** Suppose  $\lambda$  is a multicomposition of  $m$  and  $\mu$  is a composition. If there exists a multicomposition  $\gamma$  such that  $\lambda$  and  $\mu$  are cut part and remaining part of  $\gamma$  associated  $m$ , we write  $\gamma = \lambda \vee \mu$  and say  $\gamma$  is the **concatenation** of  $\lambda$  and  $\mu$ .

For example, suppose  $\lambda = (2^2, 1|3^3|2)$  and  $\mu = (4, 2)$ , then  $\gamma = \lambda \vee \mu = (2^2, 1|3^3|2, 4, 2)$ . Notice that in general  $\gamma$  is not a multipartition.

The following Corollaries follows by the definition of  $\lambda \vee \mu$ .

**Corollary 3.35.** Suppose  $\lambda$  is a multipartition of  $n$  and  $\mu, \gamma$  are partitions of  $m$ . Then  $\mu > \gamma$  if and only if  $\lambda \vee \mu > \lambda \vee \gamma$ .

**Corollary 3.36.** Suppose  $\lambda$  is a multipartition of  $n$  and  $\mu$  is a partition of  $m$ . If  $\gamma = \lambda \vee \mu$ ,  $\hat{\theta}_{i_\lambda}(\hat{e}_\mu \hat{y}_\mu)_{y_\lambda} = e_\gamma y_\gamma$ .

**Corollary 3.37.** Suppose  $\lambda$  and  $\mu$  are multipartitions and  $\gamma$  is a partition such that  $\lambda = \mu \vee \gamma$ . If  $\mathbf{u}$  and  $\mathbf{v}$  are standard  $\gamma$ -tableaux, there exist standard  $\lambda$ -tableaux  $\hat{\mathbf{u}}$  and  $\hat{\mathbf{v}}$  such that  $\hat{\theta}_{i_\mu}(\hat{\psi}_{\mathbf{uv}})_{y_\mu} = \psi_{\hat{\mathbf{u}}\hat{\mathbf{v}}}$ .

*Proof.* Suppose  $\lambda \in \mathcal{P}_n^\Lambda$  and  $\mu \in \mathcal{P}_m^\Lambda$ . By Definition 3.32,  $\mu$  is the cut part of  $\lambda$  associated to  $m$ . Let  $a$  be the cut row associated to  $m$ . Define  $\hat{u}$  to be the standard  $\lambda$ -tableau such that  $\hat{u}|_m = \mathbf{t}^\mu$ , and for any  $k > m$ , if  $\hat{u}^{-1}(k) = (r_1, c_1, \ell_1)$  and  $\hat{u}^{-1}(k-m) = (r_2, c_2, 1)$ , then

$$\begin{aligned} c_1 &= c_2, \\ r_1 &= r_2 + a. \end{aligned}$$

Define  $\hat{v}$  similarly. It is trivial that  $\hat{\theta}_{\hat{u}}(\hat{\psi}_{d(\hat{u})}) = \psi_{d(\hat{u})}$  and  $\hat{\theta}_{\hat{v}}(\hat{\psi}_{d(\hat{v})}) = \psi_{d(\hat{v})}$ . Therefore by Corollary 3.36,

$$\hat{\theta}_{\hat{u}}(\hat{\psi}_{uv}) = \hat{\theta}_{\hat{u}}(\hat{\psi}_{d(\hat{u})}^*) \hat{\theta}_{\hat{v}}(\hat{\psi}_{d(\hat{v})}) \hat{\theta}_{\hat{u}}(\hat{\psi}_{d(\hat{v})}) = \psi_{d(\hat{u})}^* e_{\lambda} \psi_{d(\hat{v})} = \psi_{\hat{u}\hat{v}}.$$

□

**Lemma 3.38.** Suppose  $\lambda \in \mathcal{P}_n^\Lambda$  and  $\mu \in \mathcal{C}_n^\Lambda$  with  $\mu > \lambda$ . If  $\mu_- \neq \lambda_-$ , then  $e_\mu y_\mu \in R_n^{>\lambda}$ .

*Proof.* As  $\mu > \lambda$  and  $\mu_- \neq \lambda_-$ , there exists  $m < n$  such that  $\mu|_m > \lambda|_m$  and  $\mu|_{m-1} = \lambda|_{m-1}$ . Set  $\nu = \mu|_m$ . If  $\nu \in \mathcal{P}_m^\Lambda$ , we have  $e_\nu y_\nu = \psi_{\mathbf{t}^\nu} \in R_n^{>\lambda|_m}$ , so by Lemma 3.14 we have  $e_\mu y_\mu \in R_n^{>\lambda}$ .

If  $\nu \notin \mathcal{P}_m^\Lambda$ , because  $\lambda \in \mathcal{P}_n^\Lambda$  and  $|\nu| = m < n$ , we have  $\nu \in \mathcal{P}_I^\Lambda$ . Notice that if we write  $\nu = (\nu^{(1)}, \dots, \nu^{(l)}, \emptyset, \dots, \emptyset)$  with  $\nu^{(l)} = (\nu_1^{(l)}, \dots, \nu_{k-1}^{(l)}, \nu_k^{(l)})$ , because  $\nu|_{m-1} = \mu|_{m-1} = \lambda|_{m-1} \in \mathcal{P}_{m-1}^\Lambda$  and  $\nu \notin \mathcal{P}_m^\Lambda$ , we must have  $\nu_{k-1}^{(l)} + 1 = \nu_k^{(l)}$ . Therefore if we write  $\mathbf{i}_\nu = (i_1, i_2, \dots, i_m)$ , we have

$$e_\nu y_\nu = e(\mathbf{i}_{\nu_-} \vee i_m) y_{\nu_-} y_m^{b_{i_m}^{\nu_-}} \in R_n^{>\nu} \subseteq R_n^{>\lambda|_m}.$$

Then by Lemma 3.14, we have  $e_\mu y_\mu \in R_n^{>\lambda}$ . This completes the proof. □

Now we are ready to start proving the main result of this section. We start by proving two more specialized Propositions. After that we will introduce a Proposition which removes these restrictions and leads to the main Theorem of this section.

**Proposition 3.39.** Suppose  $\lambda \in \mathcal{P}_n^\Lambda$  and  $\lambda_- = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  with  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_{l-1}^{(\ell)}, \lambda_l^{(\ell)}) \neq \emptyset$ . Write  $\mathbf{i}_{\lambda_-} = (i_1, i_2, \dots, i_{n-1})$ . For  $k \in I$ , if  $k \neq i_{n-1} + 1$ , or  $k = i_{n-1} + 1$  and  $\lambda_{l-1}^{(\ell)} > \lambda_l^{(\ell)}$ , we have

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

*Proof.* For convenience set  $m = \lambda_l^{(\ell)}$  and  $\mu = \lambda|_{n-m-1}$ . Therefore  $\mu = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)}, \mu^{(\ell)})$  where

$$\mu^{(\ell)} = \begin{cases} (\lambda_1^{(\ell)}, \dots, \lambda_{l-1}^{(\ell)}), & \text{if } l > 1, \\ \emptyset, & \text{if } l = 1. \end{cases}$$

Suppose  $i$  is the residue of node  $(l, 1, \ell)$  in  $\lambda$  and  $\Lambda' = \Lambda_i$ . Define  $\gamma = (m+1) \in \mathcal{P}_{m+1}^{\Lambda'}$ . Notice that  $\lambda_- = \mu \vee \gamma_-$ . Because  $k \neq i_{n-1} + 1$  or  $k = i_{n-1} + 1$  and  $\lambda_{l-1}^{(\ell)} > \lambda_l^{(\ell)}$ , we have  $b_k^{\gamma_-} = b_k^{\lambda_-}$ . By Lemma 3.9, in  $\mathcal{R}_{m+1}^{\Lambda'}$  we have  $e(\mathbf{i}_{\gamma_-} \vee k) y_{\gamma_-} y_{m+1}^{b_k^{\gamma_-}} \in R_{m+1}^{>\gamma}$ . This implies that in  $\mathcal{R}_{m+1}$ ,  $\hat{e}(\mathbf{i}_{\gamma_-} \vee k) \hat{y}_{\gamma_-} \hat{y}_{m+1}^{b_k^{\gamma_-}} \in N_{m+1}^{\Lambda'}$ . Then let  $\hat{\theta}_{\hat{u}} : \mathcal{R}_{m+1} \rightarrow \mathcal{R}_n^\Lambda$ , by Lemma 3.33,

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = e(\mathbf{i}_\mu \vee \mathbf{i}_{\gamma_-} \vee k) y_{\lambda_-} y_n^{b_k^{\gamma_-}} = \hat{\theta}_{\hat{u}}(\hat{e}(\mathbf{i}_{\gamma_-} \vee k) \hat{y}_{\gamma_-} \hat{y}_{m+1}^{b_k^{\gamma_-}}) y_\mu \in \hat{\theta}_{\hat{u}}(N_{m+1}^{\Lambda'}) y_\mu \subseteq R_n^{>\lambda},$$

which completes the proof. □

**Proposition 3.40.** Suppose  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathcal{P}_n^\Lambda$  with  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_{l-1}^{(\ell)}, \lambda_l^{(\ell)}, 1)$  and  $l \geq 2$ , where  $\lambda_{l-1}^{(\ell)} = \lambda_l^{(\ell)}$ . Write  $\mathbf{i}_{\lambda_-} = (i_1, i_2, \dots, i_{n-1})$ . Suppose  $k \in I$  and  $k \equiv i_{n-1} + 1 \pmod{e}$ . Then

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

*Proof.* For convenience set  $m = \lambda_{l-1}^{(\ell)} = \lambda_l^{(\ell)}$ , and  $\mu = \lambda|_{n-2m-1}$ . Therefore  $\mu = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)}, \mu^{(\ell)})$  where

$$\mu^{(\ell)} = \begin{cases} (\lambda_1^{(\ell)}, \dots, \lambda_{l-2}^{(\ell)}), & \text{if } l > 2, \\ \emptyset, & \text{if } l = 2. \end{cases}$$

Suppose  $i$  is the residue of node  $(l-1, 1, \ell)$  in  $\lambda$  and  $\Lambda' = \Lambda_i$ . Define  $\gamma = (m, m+1) \in \mathcal{P}_{2m+1}^{\Lambda'}$ . Notice that  $\lambda_- = \mu \vee \gamma_-$ . Because  $k \equiv i_{n-1} + 1 \pmod{e}$ , we have  $b_k^{\gamma_-} = b_k^{\lambda_-}$  and  $e(\mathbf{i}_{\gamma_-} \vee k) y_{\gamma_-} y_{2m+1}^{b_k^{\gamma_-}} = e_\gamma y_\gamma$ . By Proposition 3.29, we have  $e_\gamma y_\gamma \in R_n^{>\gamma}$ . Therefore we can write  $e_\gamma y_\gamma = \sum_{\substack{\mathbf{u}, \mathbf{v} \in \text{Std}(\sigma) \\ \sigma > \gamma}} c_{\mathbf{u}\mathbf{v}} \psi_{\mathbf{u}\mathbf{v}}$  with  $\sigma = (\sigma_1, \sigma_2)$  where  $\sigma_2 \geq 0$  and  $\sigma_1 > \gamma_1 = m$ . Therefore in  $\mathcal{R}_{2m+1}$ , we have

$$\hat{e}_\gamma \hat{y}_\gamma = \sum_{\substack{\mathbf{u}, \mathbf{v} \in \text{Std}(\sigma) \\ \sigma > \gamma}} c_{\mathbf{u}\mathbf{v}} \hat{\psi}_{\mathbf{u}\mathbf{v}} + r,$$

with  $r \in N_{2m+1}^{\Lambda'}$  and  $c_{uv} \in \mathbb{Z}$ . Therefore

$$\begin{aligned} e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} &= e(\mathbf{i}_{\mu} \vee \mathbf{i}_{\gamma_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{e}(\mathbf{i}_{\gamma_-} \vee k) \hat{\gamma}_{\gamma_-} \hat{\gamma}_{2m+1}^{b_k^{\gamma_-}}) y_{\mu} = \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{e}_{\gamma} \hat{\gamma}_{\gamma}) y_{\mu} \\ &= \sum_{\substack{u, v \in \text{Std}(\sigma) \\ \sigma > \gamma}} c_{uv} \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{uv}) y_{\mu} + \hat{\theta}_{\mathbf{i}_{\mu}}(r). \end{aligned} \quad (3.41)$$

For the first term of (3.41), define  $\alpha = \mu \vee \sigma \in \mathcal{C}_n^{\Lambda}$ . Because  $\sigma > \gamma$ , by Corollary 3.35 we have  $\alpha = \mu \vee \sigma > \mu \vee \gamma > \lambda$ . Therefore

$$\hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{uv}) y_{\mu} = \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{d(u)}^*) \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{e}_{\sigma} \hat{\gamma}_{\sigma}) y_{\mu} \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{d(v)}) = \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{d(u)}^*) e_{\mu \vee \sigma} y_{\mu \vee \sigma} \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{d(v)}) = \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{d(u)}^*) e_{\alpha} y_{\alpha} \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{d(v)})$$

Because  $\sigma = (\sigma_1, \sigma_2) > \gamma = (m, m+1)$ , we must have  $\sigma_1 > m$ . Therefore  $\alpha_- = \mu \vee \sigma_- \neq \mu \vee \gamma_- = \lambda_-$ . Then by Lemma 3.38,  $e_{\alpha} y_{\alpha} \in R_n^{>\lambda}$ . By Lemma 3.12, we have  $\hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{uv}) y_{\mu} = \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{d(u)}^*) e_{\alpha} y_{\alpha} \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{d(v)}) \in R_n^{>\lambda}$  which yields  $\sum_{\substack{u, v \in \text{Std}(\sigma) \\ \sigma > \gamma}} c_{uv} \hat{\theta}_{\mathbf{i}_{\mu}}(\hat{\psi}_{uv}) y_{\mu} \in R_n^{>\lambda}$ .

For the second term of (3.41), by Lemma 3.33, we have  $\hat{\theta}_{\mathbf{i}_{\mu}}(r) \in R_n^{>\lambda}$ . Therefore

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

□

Suppose  $\lambda \in \mathcal{S}_n^{\Lambda}$ . If  $\lambda_- = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell-1)}, \lambda^{(\ell)})$  with  $\lambda^{(\ell)} \neq \emptyset$  by Proposition 3.39 and Proposition 3.40 we have  $\lambda \in \mathcal{P}_I^{\Lambda}$ . In the rest of the subsection we are going to prove the result is still true if  $\lambda^{(\ell)} = \emptyset$ .

Suppose  $\mu = (\mu^{(1)}, \dots, \mu^{(\ell)}) \in \mathcal{P}_n^{\Lambda}$  and  $\kappa_{\Lambda} = (\kappa_1, \dots, \kappa_{\ell})$ , if  $\mu^{(\ell)} = \emptyset$ , we define  $\bar{\Lambda} = \Lambda - \Lambda_{\kappa_{\ell}}$ ,  $\kappa_{\bar{\Lambda}} = (\kappa_1, \dots, \kappa_{\ell-1})$  and  $\bar{\mu} = (\mu^{(1)}, \dots, \mu^{(\ell-1)}) \in \mathcal{P}_n^{\bar{\Lambda}}$ . Suppose  $u, v$  are two standard  $\mu$ -tableaux, define  $\bar{u}$  and  $\bar{v}$  to be standard  $\bar{\mu}$ -tableaux obtained by removing the  $\emptyset$  at the end of  $u$  and  $v$  respectively. Write  $k = \kappa_{\ell}$  for convenience. If  $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ , define

$$y_{\mathbf{i}, k} = y_1^{\delta_{i_1, k}} y_2^{\delta_{i_2, k}} \dots y_n^{\delta_{i_n, k}}.$$

**Lemma 3.42.** Suppose the notations are defined as above and  $\mathbf{i}_v$  is the residue sequence of  $v$ , then

$$\psi_{uv} = \psi_{\bar{u}\bar{v}} y_{\mathbf{i}_v, k},$$

where  $\psi_{uv}$  is an element in  $\mathcal{R}_n^{\Lambda}$  and  $\psi_{\bar{u}\bar{v}}$  is an element in  $\mathcal{R}_n^{\bar{\Lambda}}$ .

*Proof.* Without loss of generality, assume  $u = \mathbf{t}^{\mu}$ . By the definition of  $\mu$  and  $\bar{\mu}$ , writing  $\mathbf{i}_{\mu} = (i_1, i_2, \dots, i_n)$ , we have  $y_{\mu} = y_{\bar{\mu}} y_1^{\delta_{i_1, k}} y_2^{\delta_{i_2, k}} \dots y_n^{\delta_{i_n, k}} = y_{\bar{\mu}} y_{\mathbf{i}_{\mu}, k}$ .

Now for any residue sequence  $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$  and any  $r$ , If  $i_r \neq i_{r+1}$

$$\begin{aligned} e(\mathbf{i}) y_1^{\delta_{i_1, k}} y_2^{\delta_{i_2, k}} \dots y_n^{\delta_{i_n, k}} \psi_r &= (e(\mathbf{i}) y_r^{\delta_{i_r, k}} y_{r+1}^{\delta_{i_{r+1}, k}} \psi_r) y_1^{\delta_{i_1, k}} \dots y_{r-1}^{\delta_{i_{r-1}, k}} y_{r+2}^{\delta_{i_{r+2}, k}} \dots y_n^{\delta_{i_n, k}} \\ &= e(\mathbf{i}) \psi_r y_r^{\delta_{i_r, k}} y_{r+1}^{\delta_{i_{r+1}, k}} y_1^{\delta_{i_1, k}} \dots y_{r-1}^{\delta_{i_{r-1}, k}} y_{r+2}^{\delta_{i_{r+2}, k}} \dots y_n^{\delta_{i_n, k}} \\ &= e(\mathbf{i}) \psi_r y_1^{\delta_{sr(i_1), k}} \dots y_n^{\delta_{sr(i_n), k}} = e(\mathbf{i}) \psi_r y_{\mathbf{i}, sr, k}. \end{aligned}$$

If  $i_r = i_{r+1}$ , then by relation (2.13), as  $\delta_{i_r, k} = \delta_{i_{r+1}, k}$ , we have the same result.

Hence

$$\begin{aligned} \psi_{uv} &= e_{\mu} y_{\mu} \psi_{d(v)} = e(\mathbf{i}_{\mu}) y_{\bar{\mu}} y_{\mathbf{i}_{\mu}, k} \psi_{d(v)} \\ &= e(\mathbf{i}_{\mu}) y_{\bar{\mu}} \psi_{d(v)} y_{\mathbf{i}_{\mu}, d(v), k} = e_{\mu} y_{\bar{\mu}} \psi_{d(v)} y_{\mathbf{i}_v, k}. \end{aligned}$$

As  $e_{\mu} = e_{\bar{\mu}}$  and  $\psi_{d(v)} = \psi_{d(\bar{v})}$ , this completes the proof. □

**Proposition 3.43.** Suppose  $\mu$ ,  $\kappa_{\Lambda}$ ,  $\bar{\mu}$ , and  $\kappa_{\bar{\Lambda}}$  are defined as above. Then  $\bar{\mu} \in \mathcal{P}_I^{\bar{\Lambda}} \cap \mathcal{P}_y^{\bar{\Lambda}} \cap \mathcal{P}_{\psi}^{\bar{\Lambda}}$  implies  $\mu \in \mathcal{P}_I^{\Lambda} \cap \mathcal{P}_y^{\Lambda} \cap \mathcal{P}_{\psi}^{\Lambda}$ .

*Proof.* We are only going to prove that  $\bar{\mu} \in \mathcal{P}_I^{\bar{\Lambda}}$  implies  $\mu \in \mathcal{P}_I^{\Lambda}$ . The other two cases are similar.

Suppose  $\bar{\mu} \in \mathcal{P}_I^{\bar{\Lambda}}$ . Then for any  $s \in I$ , by the definition of  $\mathcal{P}_I^{\bar{\Lambda}}$ ,

$$e(\mathbf{i}_{\bar{\mu}} \vee s) y_{\bar{\mu}} y_n^{b_s^{\bar{\mu}}} = \sum_{\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})} c_{\bar{u}\bar{v}} \psi_{\bar{u}\bar{v}},$$

where  $\mathbf{i}_{\bar{v}} = \mathbf{i}_{\bar{\mu}} \vee s = \mathbf{i}_{\mu} \vee s$  and  $c_{\bar{u}\bar{v}} \in \mathbb{Z}$ .

Also we have  $e(\mathbf{i}_{\bar{\mu}} \vee s) y_{\bar{\mu}} y_n^{b_s^{\bar{\mu}}} = \theta_s(\psi_{\bar{\mu} - \bar{\mu} - s}) y_n^{b_s^{\bar{\mu}}}$ . Therefore we have

$$\theta_s(\psi_{\bar{\mu} - \bar{\mu} - s}) y_n^{b_s^{\bar{\mu}}} = \sum_{\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})} c_{\bar{u}\bar{v}} \psi_{\bar{u}\bar{v}}.$$

Notice that  $\bar{t}^{\mu_-} = \overline{t^{\mu_-}}$ . Recall  $k = \kappa_\ell$ , the last term of the multicharge  $\kappa_\Lambda$ . We consider two cases,  $s \neq k$  and  $s = k$  in the rest of the proof.

If  $s \neq k$ , then  $b_s^{\mu_-} = b_s^{\bar{\mu}_-}$ . Hence by Lemma 3.42

$$\begin{aligned} e(\mathbf{i}_{\mu_-} \vee s)y_{\mu_-}y_n^{b_s^{\mu_-}} &= \theta_s(\psi_{\mu_- - \mu_-})y_n^{b_s^{\mu_-}} \\ &= \theta_s(\psi_{\bar{\mu}_- - \bar{\mu}_-}y_{\mathbf{i}_{\mu_-},k})y_n^{b_s^{\mu_-}} \\ &= \theta_s(\psi_{\bar{\mu}_- - \bar{\mu}_-})y_n^{b_s^{\mu_-}}y_{\mathbf{i}_{\mu_-},k} \\ &= \sum_{\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})} c_{\bar{u}\bar{v}}\psi_{\bar{u}\bar{v}}y_{\mathbf{i}_{\mu_-},k}, \end{aligned}$$

and as  $s \neq k$ ,  $\delta_{s,k} = 0$ . Hence  $y_{\mathbf{i}_{\mu_-},k} = y_{\mathbf{i}_{\mu_-} \vee s,k} = y_{\mathbf{i}_q,k} = y_{\mathbf{i}_v,k}$ . By Lemma 3.42,

$$e(\mathbf{i}_{\mu_-} \vee s)y_{\mu_-}y_n^{b_s^{\mu_-}} = \sum_{\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})} c_{\bar{u}\bar{v}}\psi_{\bar{u}\bar{v}}y_{\mathbf{i}_v,k} = \sum_{\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})} c_{\bar{u}\bar{v}}\psi_{uv} \in R_n^{>\lambda},$$

because  $\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})$  implies  $u, v \in \text{Std}(>\mu)$ .

If  $s = k$ , then  $b_s^{\mu_-} = b_s^{\bar{\mu}_-} + 1$ . Hence by Lemma 3.42

$$\begin{aligned} e(\mathbf{i}_{\mu_-} \vee s)y_{\mu_-}y_n^{b_s^{\mu_-}} &= \theta_s(\psi_{\mu_- - \mu_-})y_n^{b_s^{\mu_-}} \\ &= \theta_s(\psi_{\bar{\mu}_- - \bar{\mu}_-}y_{\mathbf{i}_{\mu_-},k})y_n^{b_s^{\mu_-}}y_n \\ &= \theta_s(\psi_{\bar{\mu}_- - \bar{\mu}_-})y_n^{b_s^{\mu_-}}y_{\mathbf{i}_{\mu_-},k}y_n \\ &= \sum_{\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})} c_{\bar{u}\bar{v}}\psi_{\bar{u}\bar{v}}y_{\mathbf{i}_{\mu_-},k}y_n, \end{aligned}$$

and as  $s = k$ ,  $\delta_{s,k} = 1$ . Hence  $y_{\mathbf{i}_{\mu_-},k}y_n = y_{\mathbf{i}_{\mu_-} \vee s,k} = y_{\mathbf{i}_q,k} = y_{\mathbf{i}_v,k}$ . By Lemma 3.42

$$e(\mathbf{i}_{\mu_-} \vee s)y_{\mu_-}y_n^{b_s^{\mu_-}} = \sum_{\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})} c_{\bar{u}\bar{v}}\psi_{\bar{u}\bar{v}}y_{\mathbf{i}_v,k} = \sum_{\bar{u}, \bar{v} \in \text{Std}(>\bar{\mu})} c_{\bar{u}\bar{v}}\psi_{uv} \in R_n^{>\lambda}.$$

These implies that  $\mu \in \mathcal{P}_I$ . □

Now we are ready to prove Theorem 3.8.

**Proof of Theorem 3.8.** Write  $\mu = \lambda_- = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell-1)}, \mu^{(\ell)})$ . If  $\mu^{(\ell)} \neq \emptyset$ , by Proposition 3.39 and Proposition 3.40, we have  $\lambda \in \mathcal{P}_I^\Lambda$ .

If  $\mu^{(\ell)} = \emptyset$ , write  $\lambda^{(\ell-1)} = (\lambda_1^{(\ell-1)}, \lambda_2^{(\ell-1)}, \dots, \lambda_{k_{\ell-1}}^{(\ell-1)})$  and define  $\gamma = (\lambda^{(1)}, \dots, \lambda^{(\ell-2)}, \gamma^{(\ell-1)}, \emptyset) \in \mathcal{P}_n^\Lambda$  with  $\gamma^{(\ell-1)} = (\lambda_1^{(\ell-1)}, \lambda_2^{(\ell-1)}, \dots, \lambda_{k_{\ell-1}}^{(\ell-1)}, 1)$  and  $\bar{\gamma} = (\lambda^{(1)}, \dots, \lambda^{(\ell-2)}, \gamma^{(\ell-1)})$ . As  $l(\bar{\gamma}) < l(\lambda) = \ell$ , by the definition of  $\mathcal{S}_n^\Lambda$ ,  $\bar{\gamma} \in \mathcal{P}_I^{\bar{\Lambda}}$ . Then by Proposition 3.43 we have  $\gamma \in \mathcal{P}_I^\Lambda$ . Since  $\gamma_- = \mu = \lambda_-$  and  $\gamma > \lambda$ , for any  $k \in I$ ,

$$e(\mathbf{i}_{\lambda_-} \vee k)y_{\lambda_-}y_n^{b_k^{\lambda_-}} = e(\mathbf{i}_{\gamma_-} \vee k)y_{\gamma_-}y_n^{b_k^{\gamma_-}} \in R_n^{>\gamma} \subseteq R_n^{>\lambda},$$

which yields that  $\lambda \in \mathcal{P}_I$ . This completes the proof.

The following Corollary is directly implied by Theorem 3.8. It will contribute to proving the  $\psi$ -problem.

**Corollary 3.44.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$  and  $\mu \in \mathcal{C}_n^\Lambda$  where  $\mu > \lambda$ . Then we have  $e_\mu y_\mu \in R_n^{>\lambda}$ .

*Proof.* If  $\mu \neq \lambda_-$ , using Lemma 3.38 we have  $e_\mu y_\mu \in R_n^{>\lambda}$ . Suppose then that  $\mu_- = \lambda_-$ . If  $\mu \in \mathcal{P}_n^\Lambda$ , then  $e_\mu y_\mu = \psi_{\nu\psi} \in R_n^{>\lambda}$ . Finally, suppose that  $\mu \notin \mathcal{P}_n^\Lambda$ . If we write  $\mu = (\mu^{(1)}, \dots, \mu^{(l)}, \emptyset, \dots, \emptyset)$  with  $\mu^{(l)} = (\mu_1^{(l)}, \dots, \mu_{k-1}^{(l)}, \mu_k^{(l)})$ , we must have  $\mu_{k-1}^{(l)} + 1 = \mu_k^{(l)}$ . If we write  $\mathbf{i}_\mu = (i_1, i_2, \dots, i_n)$ , we have  $e_\mu y_\mu = e(\mathbf{i}_{\mu_-} \vee i_n)y_{\mu_-}y_n^{b_{i_n}^{\mu_-}}$ . By Theorem 3.8, as  $\lambda \in \mathcal{S}_n^\Lambda$ , we have  $\lambda \in \mathcal{P}_I^\Lambda$ . Since  $\lambda_- = \mu_-$ ,

$$e_\mu y_\mu = e(\mathbf{i}_{\mu_-} \vee i_n)y_{\mu_-}y_n^{b_{i_n}^{\mu_-}} = e(\mathbf{i}_{\lambda_-} \vee i_n)y_{\lambda_-}y_n^{b_{i_n}^{\lambda_-}} \in R_n^{>\lambda}.$$

□

## 4. Integral Basis Theorem II

In this section our main purpose is to prove that  $\mathcal{R}_n^\Lambda = R_n^\Lambda$  by proving that  $\psi_{\text{st}y_r}$  and  $\psi_{\text{st}}\psi_r$  are both in  $R_n^\Lambda$ . We first define an integer  $m_\lambda$  such that if  $\mathbf{t} \in \text{Std}(\lambda)$  and  $l(d(\mathbf{t})) < m_\lambda$ , we have  $\psi_{\text{st}y_r} \in R_n^\Lambda$  and  $\psi_{\text{st}}\psi_r \in R_n^\Lambda$ . Our first step is to show that  $m_\lambda > 0$ . Then we prove if  $l(d(\mathbf{t})) \leq m_\lambda$ , we will have  $\psi_{\text{st}y_r} \in R_n^\Lambda$  and  $\psi_{\text{st}}\psi_r \in R_n^\Lambda$  as well. By induction we will show that for any  $\mathbf{t} \in \text{Std}(\lambda)$ ,  $l(d(\mathbf{t})) < m_\lambda$ , which indicates that  $\psi_{\text{st}y_r} \in R_n^\Lambda$  and  $\psi_{\text{st}}\psi_r \in R_n^\Lambda$  for any  $\mathbf{s}, \mathbf{t} \in \text{Std}(\lambda)$ . Finally combining the results from the last section, we can prove that  $\mathcal{R}_n^\Lambda = R_n^\Lambda$ .

### 4.1. Base case of induction

In this section we fix  $\lambda \in \mathcal{S}_n^\Lambda$ . First we will give a proper definition of  $m_\lambda$ .

**Definition 4.1.** Define  $m_\lambda$  to be the smallest nonnegative integer such that for any standard  $\lambda$ -tableau  $\mathbf{t}$  with  $l(d(\mathbf{t})) < m_\lambda$  we have

$$\begin{aligned} \psi_{\text{st}y_r} &= \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})} c_{\mathbf{uv}} \psi_{\mathbf{uv}}, \\ \psi_{\text{st}}\psi_r &= \begin{cases} \psi_{\mathbf{sw}} + \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})} c_{\mathbf{uv}} \psi_{\mathbf{uv}}, & \text{if } \mathbf{w} = \mathbf{t} \cdot s_r \text{ is standard and } d(\mathbf{u}) \cdot s_r \text{ is reduced,} \\ \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})} c_{\mathbf{uv}} \psi_{\mathbf{uv}}, & \text{if } \mathbf{u} \cdot s_r \text{ is not standard or } d(\mathbf{u}) \cdot s_r \text{ is not reduced.} \end{cases} \end{aligned}$$

for some  $c_{\mathbf{uv}} \in \mathbb{Z}$ .

We will use induction to prove that for any  $\mathbf{t} \in \text{Std}(\lambda)$ ,  $l(d(\mathbf{t})) < m_\lambda$  in this section. In this subsection we will prove that  $m_\lambda > 0$ , which is the base case of the induction.

**Lemma 4.2.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$ . For any  $1 \leq r \leq n$ ,  $e_{\lambda} y_{\lambda} y_r = \psi_{\mathbf{t}^{\lambda} \mathbf{t}^{\lambda}} y_r \in R_n^{\geq \lambda}$ .

*Proof.* If  $r < n$ , write  $\mu = \lambda|_r$ . As  $\lambda \in \mathcal{S}_n^\Lambda$  we have  $\mu \in \mathcal{P}_y^\Lambda$ . Therefore  $e_{\mu} y_{\mu} y_r \in R_n^{\geq \mu}$ . By Lemma 3.14, we have  $e_{\lambda} y_{\lambda} y_r \in R_n^{\geq \lambda}$ .

If  $r = n$ , write  $\mathbf{i}_\lambda = (i_1, i_2, \dots, i_n)$ . There exists a positive integer  $b$  such that  $e_{\lambda} y_{\lambda} = e(\mathbf{i}_\lambda \vee i_n) y_{\lambda} y_n^b$ . By the definition of  $b_{i_n}^{\lambda}$  we have  $b < b_{i_n}^{\lambda}$ . If  $b + 1 < b_{i_n}^{\lambda}$ , by Lemma 3.3 there exists  $\nu \in \mathcal{P}_n^\Lambda$  such that

$$e_{\lambda} y_{\lambda} y_n = e(\mathbf{i}_\lambda \vee i_n) y_{\lambda} y_n^{b+1} = e_{\nu} y_{\nu},$$

and it is trivial that  $\nu > \lambda$ . Therefore  $e_{\lambda} y_{\lambda} y_n \in R_n^{\geq \lambda}$ . If we have  $b + 1 = b_{i_n}^{\lambda}$ , by Theorem 3.8 we have

$$e_{\lambda} y_{\lambda} y_n = e(\mathbf{i}_\lambda \vee i_n) y_{\lambda} y_n^{b+1} = e(\mathbf{i}_\lambda \vee i_n) y_{\lambda} y_n^{b_{i_n}^{\lambda}} \in R_n^{\geq \lambda},$$

which completes the proof.  $\square$

**Lemma 4.3.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$ . For any  $1 \leq r < n$ ,  $e_{\lambda} y_{\lambda} \psi_r = \psi_{\mathbf{t}^{\lambda} \mathbf{t}^{\lambda}} \psi_r \in R_n^{\geq \lambda}$ .

*Proof.* Suppose  $\mathbf{t}^{\lambda} \cdot s_r = \mathbf{t}$  is standard, we have  $e_{\lambda} y_{\lambda} \psi_r = \psi_{\mathbf{t}^{\lambda} \mathbf{t}^{\lambda}} \in R_n^{\geq \lambda}$ . So we consider the case that  $\mathbf{t}^{\lambda} \cdot s_r = \mathbf{t}$  is not standard. If  $r < n - 1$ , as  $\lambda \in \mathcal{S}_n^\Lambda$ , we have  $\mu = \lambda_- \in \mathcal{P}_\psi^\Lambda$ . Because  $\mathbf{t}^{\mu} \cdot s_r = \mathbf{t}|_{n-1}$  which is not standard,  $e_{\mu} y_{\mu} \psi_r \in R_n^{\geq \mu}$ . Then by Lemma 3.14, we have  $e_{\lambda} y_{\lambda} \psi_r \in R_n^{\geq \lambda}$ . If  $r = n - 1$  and write  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)}, \emptyset, \dots, \emptyset)$  with  $\lambda^{(l)} = (\lambda_1^{(l)}, \dots, \lambda_{k-1}^{(l)}, \lambda_k^{(l)})$ , we must have either  $\lambda_k^{(l)} \geq 2$  or  $\lambda_{k-1}^{(l)} = \lambda_k^{(l)} = 1$ . Then set  $\mu = (\lambda^{(1)}, \dots, \mu^{(l)}, \emptyset, \dots, \emptyset)$  with

$$\mu^{(l)} = \begin{cases} (\lambda_1^{(l)}, \dots, \lambda_{k-1}^{(l)}), & \text{if } \lambda_k^{(l)} \geq 2 \text{ and } k > 1, \\ (\lambda_1^{(l)}, \dots, \lambda_{k-2}^{(l)}), & \text{if } \lambda_{k-1}^{(l)} = \lambda_k^{(l)} = 1 \text{ and } k > 2, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Suppose  $i$  is the residue of node  $(k, 1, l)$  in  $\lambda$  or the residue of node  $(k - 1, 1, l)$  in  $\lambda$ ,  $\Lambda' = \Lambda_i$ ,  $m = \lambda_k^{(l)}$  or  $m = 2$  and  $\gamma = (m) \in \mathcal{P}_m^{\Lambda'}$  or  $\gamma = (1, 1) \in \mathcal{P}_m^{\Lambda'}$  if  $\lambda_k^{(l)} \geq 2$  or  $\lambda_{k-1}^{(l)} = \lambda_k^{(l)} = 1$  respectively. Therefore  $\lambda = \mu \vee \gamma$ . Because  $\lambda \in \mathcal{S}_n^\Lambda$ , we have  $\gamma \in \mathcal{P}_\psi^{\Lambda'}$ . Hence because  $\mathbf{t}^{\gamma} \cdot s_{m-1}$  is not standard, we have  $e_{\gamma} y_{\gamma} \psi_{m-1} \in R_n^{\geq \gamma} = N_m^{\Lambda'}$ . Then by Lemma 3.33,

$$e_{\lambda} y_{\lambda} \psi_r = e(\mathbf{i}_\mu \vee \mathbf{i}_\gamma) y_{\lambda} \psi_r = \hat{\theta}_{i_\mu} (e_{\gamma} y_{\gamma} \psi_{m-1}) y_{\mu} \in R_n^{\geq \lambda},$$

which completes the proof.  $\square$

**Corollary 4.4.** For  $\lambda \in \mathcal{S}_n^\Lambda$ , we have  $m_\lambda > 0$ .

*Proof.* Combining the above two Lemmas, Lemma 2.36 and Proposition 2.41, the Corollary follows.  $\square$

#### 4.2. Completion of the y-problem

In this subsection we are going to prove that for any  $t \in \text{Std}(\lambda)$ , if  $l(d(t)) \leq m_\lambda$ , then for any  $1 \leq r \leq n-1$  and any  $s \in \text{Std}(\lambda)$ , if  $t \cdot s_r$  is standard and  $d(t) \cdot s_r$  is reduced,  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$  and for any  $1 \leq r \leq n$ ,  $\psi_{st} y_r \in R_n^{\geq \lambda}$ .

First we introduce the following Lemma.

**Lemma 4.5.** *Suppose  $m$  is a positive integer such that  $m \leq m_\lambda$ , then*

$$e_\lambda y_\lambda \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} =_\lambda \sum_{\substack{v \in \text{Std}(\lambda) \\ l(d(v)) \leq m}} c_{t^i v} \psi_{t^i v}.$$

*Proof.* We apply induction on  $m$ . Suppose  $m = 0$  then there is nothing to prove. Assume for any  $m' < m$  the Lemma holds. Therefore  $e_\lambda y_\lambda \psi_{r_1} \psi_{r_2} \dots \psi_{r_{m-1}} =_\lambda \sum_{\substack{u \in \text{Std}(\lambda) \\ l(d(u)) \leq m-1}} c_{t^i u} \psi_{t^i u}$  which yields

$$e_\lambda y_\lambda \psi_{r_1} \psi_{r_2} \dots \psi_{r_{m-1}} \psi_{r_m} =_\lambda \sum_{\substack{u \in \text{Std}(\lambda) \\ l(d(u)) \leq m-1}} c_{t^i u} \psi_{t^i u} \psi_{r_m}.$$

For  $u \in \text{Std}(\lambda)$  and  $l(d(u)) \leq m-1 < m_\lambda$ , if  $u \cdot s_r$  is standard and  $s_{d(v)} \cdot s_{r_m}$  is reduced, by the definition of  $m_\lambda$ ,

$$\psi_{t^i u} \psi_{r_m} = \psi_{t^i, u \cdot s_r} + \sum_{(x, y) \triangleright (t^i, u)} c_{xy} \psi_{xy} =_\lambda \psi_{t^i, u \cdot s_r} + \sum_{v \triangleright u} c_{t^i v} \psi_{t^i v},$$

where  $l(d(u \cdot s_r)) = 1 + l(d(u)) \leq m$  and  $l(d(v)) < l(d(u)) < m$  as  $v \triangleright u$ . Hence

$$\psi_{t^i u} \psi_{r_m} =_\lambda \sum_{\substack{v \in \text{Std}(\lambda) \\ l(d(v)) \leq m}} c_v \psi_{t^i v}.$$

If  $u \cdot s_r$  is not standard or  $s_{d(v)} \cdot s_{r_m}$  is not reduced, we have

$$\psi_{t^i u} \psi_{r_m} = \sum_{(x, y) \triangleright (t^i, u)} c_{xy} \psi_{xy} =_\lambda \sum_{v \triangleright u} c_{t^i v} \psi_{t^i v},$$

where  $l(d(v)) < l(d(u)) \leq m-1 < m$  as  $v \triangleright u$ . Hence

$$\psi_{t^i u} \psi_{r_m} =_\lambda \sum_{\substack{v \in \text{Std}(\lambda) \\ l(d(v)) \leq m}} c_{t^i v} \psi_{t^i v},$$

which completes the proof.  $\square$

Now we can start to prove that  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$  when  $t \cdot s_r$  is standard and  $d(t) \cdot s_r$  is reduced.

**Lemma 4.6.** *Suppose  $t$  is a standard  $\lambda$ -tableau with  $d(t) = s_{r_1} s_{r_2} \dots s_{r_l}$  where  $l \leq m_\lambda$ , and  $d'(t) = s_{t_1} s_{t_2} \dots s_{t_l}$  is another reduced decomposition of  $d(t)$ , then*

$$e_\lambda y_\lambda \psi_{d(t)} - e_\lambda y_\lambda \psi_{d'(t)} = \sum_{(u, v) \triangleright (t^i, t)} c_{uv} \psi_{uv}.$$

*Proof.* By [5, Proposition 2.5], we have

$$y_\lambda e_\lambda \psi_{d(t)} - y_\lambda e_\lambda \psi_{d'(t)} = \sum_{u < d(t)} c_{u, f} y_\lambda e_\lambda f(y) \psi_u,$$

where  $f(y)$  is a polynomial in  $y_r$ 's and  $u$  is a word in  $\mathfrak{S}_n$ . If  $f(y) \neq 1$ , by Lemma 4.2 we have  $e_\lambda y_\lambda f(y) \in R_n^{\geq \lambda}$ . Hence  $y_\lambda e_\lambda f(y) \psi_u \in R_n^{\geq \lambda}$ . If  $f(y) = 1$ , as  $u < d(t)$  then  $l(u) < l \leq m_\lambda$ , by Lemma 4.5 we have  $e_\lambda y_\lambda \psi_u \in R_n^{\geq \lambda}$ . Henceforth

$$y_\lambda e_\lambda \psi_w - y_\lambda e_\lambda \psi_{w'} \in R_n^{\geq \lambda}.$$

Then by Proposition 2.41 and [9, Lemma 5.7], we complete the proof.  $\square$

The following Corollary is straightforward by Lemma 4.6 which explains the action of  $\psi_r$  to  $\psi_{st}$  from right when  $t \cdot s_r$  is standard and  $d(t) \cdot s_r$  is reduced.

**Corollary 4.7.** *Suppose  $t$  is a standard  $\lambda$ -tableau with  $l(d(t)) \leq m_\lambda$ , if  $w = t \cdot s_r$  is standard and  $d(t) \cdot s_r$  is reduced,*

$$\psi_{st} \psi_r = \psi_{sw} + \sum_{(u, v) \triangleright (s, t)} c_{uv} \psi_{uv}.$$

Now we start to prove that  $\psi_{st} y_r \in R_n^{\geq \lambda}$ .



**Lemma 4.8.** Suppose  $\mathbf{t}$  is a standard  $\lambda$ -tableau with  $l(d(\mathbf{t})) < m_\lambda$ . For any  $1 \leq k \leq n$ ,  $1 \leq r \leq n-1$  and any standard  $\lambda$ -tableau  $\mathbf{s}$ , we have

$$\psi_{\mathbf{st}} y_k \psi_r \in R_n^{\geq \lambda}.$$

*Proof.* As  $l(d(\mathbf{t})) < m_\lambda$ , we have

$$\psi_{\mathbf{st}} y_k = \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})} c_{\mathbf{uv}} \psi_{\mathbf{uv}} = \sum_{\mathbf{v} \triangleright \mathbf{t}} c_{\mathbf{sv}} \psi_{\mathbf{sv}} + \sum_{\mathbf{u}, \mathbf{v} \in \text{Std}(> \lambda)} c_{\mathbf{uv}} \psi_{\mathbf{uv}}.$$

For  $\text{Shape}(\mathbf{v}) = \lambda$ , since  $\mathbf{v} \triangleright \mathbf{t}$ ,  $l(d(\mathbf{v})) < l(d(\mathbf{t})) < m_\lambda$ . Then we have  $\psi_{\mathbf{sv}} \psi_r \in R_n^{\geq \lambda}$ .

For  $\mathbf{u}, \mathbf{v} \in \text{Std}(> \lambda)$ ,  $\psi_{\mathbf{uv}} \in R_n^{> \lambda}$ . As  $\lambda \in \mathcal{S}_n^\Lambda$ ,  $R_n^{> \lambda}$  is an ideal by Lemma 3.12. Hence  $\psi_{\mathbf{uv}} \psi_{r_l} \in R_n^{> \lambda}$  and this completes the proof.  $\square$

**Proposition 4.9.** Suppose  $\mathbf{t}$  is a standard  $\lambda$ -tableau with  $l(d(\mathbf{t})) \leq m_\lambda$ , for any  $1 \leq r \leq n$  and any standard  $\lambda$ -tableau  $\mathbf{s}$ , we have

$$\psi_{\mathbf{st}} y_r = \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})} c_{\mathbf{uv}} \psi_{\mathbf{uv}}.$$

*Proof.* Write  $d(\mathbf{t}) = s_{r_1} s_{r_2} \dots s_{r_{l-1}} s_{r_l}$  and  $\mathbf{w} = \mathbf{t} \cdot s_{r_l} = s_{r_1} s_{r_2} \dots s_{r_{l-1}}$ . We prove this Proposition by considering different values of  $r$ .

If  $r \notin \{r_l, r_l + 1\}$ , then  $\psi_{r_l}$  and  $y_r$  commute. Hence

$$\psi_{\mathbf{st}} y_r = \psi_{d(\mathbf{s})}^* e_{\lambda} y_{\lambda} \psi_{d(\mathbf{t})} y_r = \psi_{d(\mathbf{s})}^* e_{\lambda} y_{\lambda} \psi_{d(\mathbf{w})} y_r \psi_{r_l} = \psi_{\mathbf{sw}} y_r \psi_{r_l}.$$

As  $l(d(\mathbf{w})) = l(d(\mathbf{t})) - 1 < m_\lambda$ , by Lemma 4.8 we have  $\psi_{\mathbf{st}} y_r \in R_n^{\geq \lambda}$ .

If  $r = r_l$ , let  $\mathbf{j}$  be a sequence such that  $e(\mathbf{i}_\lambda) \psi_{d(\mathbf{t})} = \psi_{d(\mathbf{w})} e(\mathbf{j}) \psi_{r_l}$ . We separate this case further into  $j_{r_l} \neq j_{r_l+1}$  and  $j_{r_l} = j_{r_l+1}$ . First suppose  $j_{r_l} \neq j_{r_l+1}$ , then

$$\psi_{\mathbf{st}} y_r = \psi_{d(\mathbf{s})}^* e_{\lambda} y_{\lambda} \psi_{d(\mathbf{t})} y_r = \psi_{d(\mathbf{s})}^* e_{\lambda} y_{\lambda} \psi_{d(\mathbf{w})} y_{r+1} \psi_{r_l} = \psi_{\mathbf{sw}} y_{r+1} \psi_{r_l}.$$

Hence as  $l(d(\mathbf{w})) < m_\lambda$ , by Lemma 4.8 we have  $\psi_{\mathbf{st}} y_r \in R_n^{\geq \lambda}$  when  $j_{r_l} \neq j_{r_l+1}$ . Now suppose  $j_{r_l} = j_{r_l+1}$ , we have

$$\psi_{\mathbf{st}} y_r = \psi_{d(\mathbf{s})}^* e_{\lambda} y_{\lambda} \psi_{d(\mathbf{t})} y_r = \psi_{d(\mathbf{s})}^* e_{\lambda} y_{\lambda} \psi_{d(\mathbf{w})} + \psi_{d(\mathbf{s})}^* e_{\lambda} y_{\lambda} \psi_{d(\mathbf{w})} y_{r+1} \psi_{r_l} = \psi_{\mathbf{sw}} + \psi_{\mathbf{sw}} y_{r+1} \psi_{r_l}.$$

As  $l(d(\mathbf{w})) < m_\lambda$ , by Lemma 4.8 we have  $\psi_{\mathbf{sw}} y_{r+1} \psi_{r_l} \in R_n^{\geq \lambda}$ . As  $\psi_{\mathbf{sw}} \in R_n^{\geq \lambda}$  as well, we have  $\psi_{\mathbf{st}} y_r \in R_n^{\geq \lambda}$ . So for  $r = r_l$ , we have  $\psi_{\mathbf{st}} y_r \in R_n^{\geq \lambda}$ .

If  $r = r_l + 1$ , the method is the same as  $r = r_l$ .

Therefore in all the cases, we have  $\psi_{\mathbf{st}} y_r \in R_n^{\geq \lambda}$ . So

$$\psi_{\mathbf{st}} y_r = \sum_{\mathbf{u}, \mathbf{v} \in \text{Std}(\geq \lambda)} c_{\mathbf{uv}} \psi_{\mathbf{uv}},$$

and by Proposition 2.41 we complete the proof.  $\square$

### 4.3. Properties of $m_\lambda$

In the rest of this section we will prove that if  $\mathbf{t} \in \text{Std}(\lambda)$  and  $l(d(\mathbf{t})) \leq m_\lambda$ , then for any  $1 \leq r \leq n-1$  and any  $\mathbf{s} \in \text{Std}(\lambda)$ , we have  $\psi_{\mathbf{st}} \psi_r \in R_n^{\geq \lambda}$ . In this subsection we will give some properties for  $m_\lambda$  which will be used in proving the above argument.

**Lemma 4.10.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$ . For any permutation  $w \in \mathfrak{S}_n$  with reduced expression  $w = s_{r_1} s_{r_2} \dots s_{r_{m-1}} s_{r_m}$  and  $r = \min\{r_1, r_2, \dots, r_m\}$ , if we write

$$e_{\lambda} y_{\lambda} \psi_w = e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} = \sum_{\mathbf{u}, \mathbf{v} \in \text{Std}(\geq \lambda)} c_{\mathbf{uv}} \psi_{\mathbf{uv}},$$

then  $c_{\mathbf{uv}} \neq 0$  implies  $\mathbf{v}|_k \triangleright \mathbf{t}^{\lambda_k}$  for any  $k < r$ .

*Proof.* We prove the Lemma by induction. If  $m = 1$ , then  $r_1 = r$ .

$$e_{\lambda} y_{\lambda} \psi_r = \begin{cases} \psi_{\mathbf{t}^{\lambda_r}}, & \text{if } \mathbf{v} = \mathbf{t}^{\lambda_r} \text{ is standard,} \\ \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{t}^{\lambda}, \mathbf{t}^{\lambda_r})} c_{\mathbf{uv}} \psi_{\mathbf{uv}}, & \text{otherwise.} \end{cases}$$

If  $\mathbf{v} = \mathbf{t}^{\lambda_r}$  is standard, then by the definition of  $\mathbf{v}$ ,  $\mathbf{v}|_k = \mathbf{t}^{\lambda_k} = \mathbf{t}^{\lambda_k}$  for  $k < r$ . If it is the other case, as  $\mathbf{v} \triangleright \mathbf{t}^{\lambda}$ , then  $\mathbf{v}|_k \triangleright \mathbf{t}^{\lambda_k} = \mathbf{t}^{\lambda_k}$ .

Assume for any  $m' < m$  the Corollary holds. Then

$$e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \dots \psi_{r_{m-1}} = \sum_{u_1, v_1 \in \text{Std}(\geq \lambda)} c_{u_1 v_1} \psi_{u_1 v_1},$$

where  $v_1|_k \geq t^{\lambda_k}$  for  $k < r$ . Then

$$e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \dots \psi_{r_{m-1}} \psi_{r_m} = \sum_{u_1, v_1 \in \text{Std}(\geq \lambda)} c_{u_1 v_1} \psi_{u_1 v_1} \psi_{r_m}.$$

Since

$$\psi_{u_1 v_1} \psi_{r_m} = \begin{cases} \psi_{u_1 v} + \sum_{(u, v_2) \triangleright (u_1, v_1)} c_{u v_2} \psi_{u v_2}, & \text{if } v = v_1 \cdot s_{r_m} \text{ is standard and } d(v_1) \cdot s_{r_m} \text{ is reduced,} \\ \sum_{(u, v) \triangleright (u_1, v_1)} c_{u v} \psi_{u v}, & \text{otherwise.} \end{cases}$$

If  $v = v_1 \cdot s_{r_m}$  is standard and  $d(v_1) \cdot s_{r_m}$  is reduced, recall  $v_1|_k \geq t^{\lambda_k}$  for  $k < r$ , as  $v = v_1 \cdot s_{r_m}$ ,  $v|_k = v_1|_k \geq t^{\lambda_k}$  for  $k < r \leq r_m$ . For  $v_2 \triangleright v_1$ , we have  $v_2|_k \triangleright v_1|_k \geq t^{\lambda_k}$  for  $k < r$ .

If it is of the other case, as  $v \triangleright v_1$ ,  $v|_k \triangleright v_1|_k \geq t^{\lambda_k}$ . Therefore

$$e_{\lambda} y_{\lambda} \psi_w = e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} = \sum_{u, v \in \text{Std}(\geq \lambda)} c_{u v} \psi_{u v},$$

and  $c_{u v} \neq 0$  implies  $v|_k \geq t^{\lambda_k}$  for any  $k < r$ . This completes the proof.  $\square$

**Lemma 4.11.** Suppose  $\lambda \in \mathcal{S}_n^{\Lambda} \cap (\mathcal{P}_I^{\Lambda} \cap \mathcal{P}_y^{\Lambda} \cap \mathcal{P}_{\psi}^{\Lambda})$ . Then for any  $1 \leq r_1, r_2, \dots, r_m \leq n-1$

$$e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} = \sum_{v \in \text{Std}(\lambda)} c_{t^{\lambda} v} \psi_{t^{\lambda} v} + \sum_{\substack{u, v \in \text{Std}(> \lambda) \\ u \triangleright t^{\lambda}}} c_{u v} \psi_{u v}.$$

*Proof.* When  $m = 1$ , we have

$$e_{\lambda} y_{\lambda} \psi_{r_1} = \begin{cases} \psi_{t^{\lambda} v}, & \text{if } v = t^{\lambda} \cdot s_{r_1} \text{ is standard,} \\ \sum_{(u, v) \triangleright (t^{\lambda}, t^{\lambda})} c_{u v} \psi_{u v} = \sum_{\substack{u, v \in \text{Std}(> \lambda) \\ u \triangleright t^{\lambda}}} c_{u v} \psi_{u v}, & \text{if } v = t^{\lambda} \cdot s_{r_1} \text{ is not standard.} \end{cases}$$

which follows the Lemma.

Suppose for  $m' < m$  the Lemma holds. Then by induction

$$e_{\lambda} y_{\lambda} \psi_{r_1} \dots \psi_{r_{m-1}} \psi_{r_m} = \sum_{v_1 \in \text{Std}(\lambda)} c_{t^{\lambda} v_1} \psi_{t^{\lambda} v_1} \psi_{r_m} + \sum_{\substack{u_1, v_1 \in \text{Std}(> \lambda) \\ u_1 \triangleright t^{\lambda}}} c_{u_1 v_1} \psi_{u_1 v_1} \psi_{r_m}. \quad (4.12)$$

For  $v_1 \in \text{Std}(\lambda)$ , as  $\lambda \in \mathcal{P}_{\psi}^{\Lambda}$ ,

$$\psi_{t^{\lambda} v_1} \psi_{r_m} = \begin{cases} \psi_{t^{\lambda} v_2} + \sum_{(u_2, v_2) \triangleright (t^{\lambda}, v_1)} c_{u_2 v_2} \psi_{u_2 v_2}, & \text{if } v_2 = v_1 \cdot s_{r_m} \text{ is standard} \\ & \text{and } d(v_2) = d(v_1) \cdot s_{r_m} \text{ is reduced,} \\ \sum_{(u_2, v_2) \triangleright (t^{\lambda}, v_1)} c_{u_2 v_2} \psi_{u_2 v_2}, & \text{if } v_2 = v_1 \cdot s_{r_m} \text{ is not standard} \\ & \text{or } d(v_2) = d(v_1) \cdot s_{r_m} \text{ is not reduced.} \end{cases}$$

where in both cases, we can write

$$\psi_{t^{\lambda} v_1} \psi_{r_m} = \sum_{v_2 \in \text{Std}(\lambda)} c_{t^{\lambda} v_2} \psi_{t^{\lambda} v_2} + \sum_{\substack{u_2, v_2 \in \text{Std}(> \lambda) \\ u_2 \triangleright t^{\lambda}}} c_{u_2 v_2} \psi_{u_2 v_2}. \quad (4.13)$$

For  $u_1, u_2 \in \text{Std}(> \lambda)$ ,

$$\psi_{u_1 v_1} \psi_{r_m} = \begin{cases} \psi_{u_1 v_2} + \sum_{(u_2, v_2) \triangleright (u_1, v_1)} c_{u_2 v_2} \psi_{u_2 v_2}, & \text{if } v_2 = v_1 \cdot s_{r_m} \text{ is standard} \\ & \text{and } d(v_2) = d(v_1) \cdot s_{r_m} \text{ is reduced,} \\ \sum_{(u_2, v_2) \triangleright (u_1, v_1)} c_{u_2 v_2} \psi_{u_2 v_2}, & \text{if } v_2 = v_1 \cdot s_{r_m} \text{ is not standard} \\ & \text{or } d(v_2) = d(v_1) \cdot s_{r_m} \text{ is not reduced.} \end{cases}$$

where since  $u_1 \triangleright t^{\lambda}$ , we can always write

$$\psi_{u_1 v_1} \psi_{r_m} = \sum_{\substack{u_2, v_2 \in \text{Std}(> \lambda) \\ u_2 \triangleright t^{\lambda}}} c_{u_2 v_2} \psi_{u_2 v_2}. \quad (4.14)$$

Therefore, substitute (4.13) and (4.14) back to (4.12), we have

$$e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} = \sum_{v \in \text{Std}(\lambda)} c_{t^1 v} \psi_{t^1 v} + \sum_{\substack{u, v \in \text{Std}(>\lambda) \\ u \triangleright t^1}} c_{uv} \psi_{uv},$$

which completes the proof.  $\square$

**Lemma 4.15.** Suppose  $\lambda \in \mathcal{S}_n^{\Lambda}$  and  $r_1, r_2, \dots, r_m$  are positive integers such that  $r_1, \dots, r_m < n - 1$ . Then

$$e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} \in R_n^{\geq \lambda}.$$

*Proof.* Define  $\mu = \lambda|_{n-1}$ . As  $\lambda \in \mathcal{S}_n^{\Lambda}$ ,  $\mu \in \mathcal{S}_{n-1}^{\Lambda} \cap (\mathcal{P}_I^{\Lambda} \cap \mathcal{P}_y^{\Lambda} \cap \mathcal{P}_{\psi}^{\Lambda})$ . Define  $i \in I$  such that  $\mathbf{i}_{\lambda} = \mathbf{i}_{\mu} \vee i$ . As  $r_1, r_2, \dots, r_m < n - 1$ , we have

$$e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} = \theta_i(e_{\mu} y_{\mu} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m}),$$

where

$$e_{\mu} y_{\mu} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} = \sum_{\check{v} \in \text{Std}(\mu)} c_{\check{v} \check{v}} \psi_{\check{v} \check{v}} + \sum_{\check{u}, \check{v} \in \text{Std}(>\mu)} c_{\check{u} \check{v}} \psi_{\check{u} \check{v}}.$$

As  $\sum_{\check{u}, \check{v} \in \text{Std}(>\mu)} c_{\check{u} \check{v}} \psi_{\check{u} \check{v}} \in R_n^{>\mu} = R_n^{>\lambda|_{n-1}}$ , by Lemma 3.14,  $\theta_i(\sum_{\check{u}, \check{v} \in \text{Std}(>\mu)} c_{\check{u} \check{v}} \psi_{\check{u} \check{v}}) \in R_n^{>\lambda}$ .

For  $\check{v} \in \text{Std}(\mu) = \text{Std}(\lambda|_{n-1})$  and  $\mathbf{i}_{\mu} \vee i = \mathbf{i}_{\lambda}$ , define  $v$  to be the standard  $\lambda$ -tableau with  $v|_{n-1} = \check{v}$ . Hence  $\theta_i(\psi_{\check{v} \check{v}}) = \psi_{t^1 v}$ . Therefore

$$\theta_i\left(\sum_{\check{v} \in \text{Std}(\mu)} c_{\check{v} \check{v}} \psi_{\check{v} \check{v}}\right) = \sum_{v \in \text{Std}(\lambda)} c_{\check{v} \check{v}} \psi_{t^1 v} \in R_n^{\geq \lambda}.$$

So

$$e_{\mu} y_{\mu} \psi_{r_1} \psi_{r_2} \dots \psi_{r_m} = \theta_i\left(\sum_{\check{v} \in \text{Std}(\mu)} c_{\check{v} \check{v}} \psi_{\check{v} \check{v}}\right) + \theta_i\left(\sum_{\check{u}, \check{v} \in \text{Std}(>\mu)} c_{\check{u} \check{v}} \psi_{\check{u} \check{v}}\right) \in R_n^{\geq \lambda}.$$

$\square$

#### 4.4. Garnir tableaux

In the following subsections we will prove that  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$  for  $l(d(t)) \leq m_{\lambda}$ . Generally, if  $t \cdot s_r$  is standard and  $d(t) \cdot s_r$  is reduced or  $l(d(t)) \cdot s_r$  is not reduced, it is comparatively easy to prove that  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$ . Our main difficulty is to prove that when  $t \cdot s_r$  is not standard then  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$ . In order to prove this we consider different types of  $t$ . Among these cases the hardest part is that when  $t$  is a special kind of tableaux which is called the Garnir tableau and  $t \cdot s_r$  is not standard. In this subsection we will prove that in such case  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$ .

The method of proving the argument in this subsection is first assuming that  $\text{Shape}(t)$  is a partition of two rows, and using the similar argument we used in the last section to extend the result to general multipartitions. First we give a detailed definition of garnir tableaux.

We introduce a special kind of tableaux, the Garnir tableaux, which was first introduced by Murphy [22]. Let  $(a, b, m)$  be a node of  $\lambda$  such that  $(a + 1, b, m)$  is also a node of  $\lambda$ . The  $(a, b, m)$ -**Garnir belt** of  $\lambda$  consists of the nodes  $(a, c, m)$  for  $b \leq c \leq \lambda_a^{(m)}$  and the nodes  $(a + 1, g, m)$  for  $1 \leq g \leq b$ . For example here is a picture of the  $(2, 3, 2)$ -Garnir belt for  $\lambda = (3, 1|7, 6, 5, 2)$ .


The  $(a, b, m)$ -**Garnir tableau** of shape  $\lambda$  is the unique maximal standard  $\lambda$ -tableau with respect to the Bruhat order ( $\triangleright$ ) among the standard  $\lambda$ -tableaux which agree with  $t^1$  outside the  $(a, b, m)$ -Garnir belt. For example the following is the  $(2, 3, 2)$ -Garnir tableau for  $\lambda = (3, 1|7, 6, 5, 2)$ .

$$\left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \middle| \begin{array}{|c|c|c|c|c|c|c|} \hline 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 12 & 13 & 16 & 18 & 19 & 20 \\ \hline 14 & 15 & 17 & 21 & 22 \\ \hline 23 \\ \hline 24 \\ \hline \end{array} \right)$$

Suppose  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  and  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_k^{(\ell)})$ . Let  $(a, b, m) = (k - 1, \lambda_k^{(\ell)}, \ell)$  and  $t$  be the  $(a, b, m)$ -Garnir tableau. Let the entry in node  $(a, b, m)$  of  $t$  be  $r$ . Then  $t \cdot s_r$  is not standard.

**Definition 4.16.** Suppose  $\lambda \in \mathcal{P}_n^{\Lambda}$  with  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  and  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_k^{(\ell)})$ . If  $k \geq 2$ , and  $t$  is the  $(k - 1, \lambda_k^{(\ell)}, \ell)$ -Garnir tableau, then we call  $t$  the **last Garnir tableau** of shape  $\lambda$ , and  $r = t(k - 1, \lambda_k^{(\ell)}, \ell)$  the **last Garnir entry** of  $t$ .

For example

$$\left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \left| \begin{array}{|c|c|c|c|c|c|} \hline 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 12 & 13 & 16 & 18 & 19 & 20 & \\ \hline 14 & 15 & 17 & & & & \\ \hline \end{array} \right. \right)$$

is the last  $(2, 3, 2)$ -Garnir tableau, and

$$\left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \left| \begin{array}{|c|c|c|c|c|c|} \hline 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 12 & 13 & 16 & 18 & 19 & 20 & \\ \hline 14 & 15 & 17 & 21 & 22 & & \\ \hline 23 & 24 & & & & & \\ \hline \end{array} \right. \right)$$

is not the last one. Notice that  $t \cdot s_r$  is not standard.

Because we are going to play around with  $\psi_{d(t)}$  a lot, we introduce more detailed notation for these elements in the next Lemma.

**Lemma 4.17.** Suppose  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ . Let  $t$  be a  $(a, b, m)$ -Garnir tableau of shape  $\lambda$  and  $\lambda^{(m)} = (\lambda_1^{(m)}, \dots, \lambda_k^{(m)})$ . Suppose

$$\begin{cases} t^\lambda(a, b, m) &= l, \\ t^\lambda(a, \lambda_a^{(m)}, m) &= s, \\ t^\lambda(a+1, b, m) &= t. \end{cases}$$

Then  $l \leq s < t$ . Write  $t - s = c$ ,

$$\psi_{d(t)} = \psi_s \psi_{s+2} \dots \psi_{t-1} \cdot \psi_{s-1} \psi_s \dots \psi_{t-2} \dots \psi_{l+1} \psi_{l+2} \dots \psi_{l+c} \cdot \psi_l \psi_{l+1} \dots \psi_{l+c-2}$$

where

$$l(\psi_s \psi_{s+1} \dots \psi_{t-1}) = l(\psi_{s-1} \psi_s \dots \psi_{t-2}) = \dots = l(\psi_{l+1} \psi_{l+2} \dots \psi_{l+c}) = c$$

and

$$l(\psi_l \psi_{l+1} \dots \psi_{l+c-2}) = c - 1.$$

*Proof.* The Lemma follows by direct calculation.  $\square$

**Example 4.18.** Suppose  $\lambda = (3, 1|7, 6, 5, 2)$  and  $(a, b, m) = (2, 3, 2)$ . Let  $t$  be the  $(2, 3, 2)$ -Garnir tableau of shape  $\lambda$ . Then

$$t = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \left| \begin{array}{|c|c|c|c|c|c|} \hline 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 12 & 13 & 16 & 18 & 19 & 20 & \\ \hline 14 & 15 & 17 & 21 & 22 & & \\ \hline 23 & 24 & & & & & \\ \hline \end{array} \right. \right),$$

and

$$t^\lambda = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \left| \begin{array}{|c|c|c|c|c|c|} \hline 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 12 & 13 & 14 & 15 & 16 & 17 & \\ \hline 18 & 19 & 20 & 21 & 22 & & \\ \hline 23 & 24 & & & & & \\ \hline \end{array} \right. \right).$$

Then

$$\begin{aligned} t^\lambda(a, b, m) &= t^\lambda(2, 3, 2) = 14, \\ t^\lambda(a, \lambda_a^{(m)}, m) &= t^\lambda(2, 6, 2) = 17, \\ t^\lambda(a+1, b, m) &= t^\lambda(3, 3, 2) = 20, \end{aligned}$$

and  $c = t - s = 3$ . Therefore

$$\psi_{d(t)} = \psi_{17} \psi_{18} \psi_{19} \cdot \psi_{16} \psi_{17} \psi_{18} \cdot \psi_{15} \psi_{16} \psi_{17} \cdot \psi_{14} \psi_{15}$$

with

$$l(\psi_{17} \psi_{18} \psi_{19}) = l(\psi_{16} \psi_{17} \psi_{18}) = l(\psi_{15} \psi_{16} \psi_{17}) = 3 = c$$

and

$$l(\psi_{14} \psi_{15}) = 2 = c - 1.$$

**Remark 4.19.** For  $a \leq b - 1$ , we will write  $\psi_{a,b} = \psi_a \psi_{a+1} \psi_{a+2} \dots \psi_{b-2} \psi_{b-1}$  and  $\psi_{b,a} = \psi_{a,b}^*$  in order to simplify our notations.

Our first step is to prove that when  $\lambda$  is a partition with two rows and  $t$  is the last Garnir tableau of shape  $\lambda$  with  $r$  as its last Garnir entry, then  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$  for any  $s \in \text{Std}(\lambda)$ . We set  $\lambda = (\lambda_1, \lambda_2)$  and without loss of generality, set  $\Lambda = \Lambda_0$ . Therefore  $\lambda \in \mathcal{P}_n^\Lambda$  with  $n = \lambda_1 + \lambda_2$ . Also we set  $\mu = (\lambda_1, \lambda_2 - 1, 1)$ ,  $\tilde{\lambda} = (\lambda_1 - 1, \lambda_2)$  and  $\hat{\mu} = (\lambda_1 - 1, \lambda_2 - 1, 1)$ . Furthermore, let  $i = \text{res}(\gamma_1)$ ,  $j = \text{res}(\gamma_2)$ , where  $\gamma_1 = (1, \lambda_1, 1)$  and  $\gamma_2 = (2, \lambda_2, 1)$ .

First we prove a few useful Lemmas.

$$e_{\lambda} y_{\lambda} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} =_{\lambda} \begin{cases} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} y_n \\ \quad - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}}, & \text{if } i = e-1, j \neq e-1, \\ \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} y_n \\ \quad - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} \\ \quad - \psi_{\lambda_1} \dots \psi_{n-2} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}}, & \text{if } i = j = e-1, \\ \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} \\ \quad + \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} e(\mathbf{i}_{\tilde{\mu}} \vee i) y_{\tilde{\mu}}, & \text{if } i = j = e-2, \\ \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}}, & \text{otherwise.} \end{cases}$$

By using the diagrammatic notation we have

$$e_{\lambda} y_{\lambda} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} =$$

**Case 4.20a:**  $i \neq e - 2, e - 1$ .

Because we set  $e = 4$  so in this case we have  $i \neq 2, 3$ . As  $i \neq 3$ ,  $\delta_{i,3} = 0$ . Therefore there are no dots on the strand labelled by  $i$ . And as  $i \neq 2$ , by relation 2.10, we have

$$\begin{aligned}
& e_{\lambda} y_{\lambda} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} \\
&= \text{Diagram 1} = \text{Diagram 2} \\
&= \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\lambda} \vee i) y_{\lambda}.
\end{aligned}$$

**Case 4.20b:**  $i = e - 1$  and  $j \neq e - 1$ .

Because we set  $e = 4$  so in this case we have  $i = 3$  and  $j \neq 3$ . Then  $\delta_{i,3} = 1$ . Hence

$$\begin{aligned}
 & e_{\lambda} y_{\lambda} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} = \\
 & \begin{array}{c}
 \text{(2.10)} \quad \text{Diagram 1} - \text{Diagram 2} \\
 - \text{Diagram 3} - \dots + \text{Diagram 4} \\
 \text{(2.17)} \quad \text{Diagram 5} - \text{Diagram 6} \\
 + \text{Diagram 7} + \dots \text{by Lemma 3.14} \\
 + \text{Diagram 8} \\
 \\
 =_{\lambda} \quad \text{Diagram 9} + \text{Diagram 10} \\
 \text{(2.10)} \quad \text{Diagram 11} + \text{Diagram 12} \\
 = -\psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\lambda} \vee i) y_{\lambda} + \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\lambda} \vee i) y_{\lambda} y_n.
 \end{array}
 \end{aligned}$$

The diagrams are represented by vertical lines with dots and crossings. The top diagram has a sequence of dots labeled 0, 1, 2, 3, 0, 2, 3, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0, j-1, j. A blue line crosses from the 3rd dot to the j-th dot. The subsequent diagrams show various transformations of this initial state, including crossings and the application of Lemma 3.14 to a specific sub-diagram.

**Case 4.20c:**  $i = j = e - 1$ .

Because we set  $e = 4$  so in this case we have  $i = j = 3$ . Similarly as in Case 4.20b, we have

$$\begin{aligned}
 & e_{\lambda} y_{\lambda} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} \\
 = & \quad - \quad \begin{array}{c} \lambda_1 \\ \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \quad - \quad \begin{array}{c} \lambda_1 \\ \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \\
 & \quad - \quad \begin{array}{c} \lambda_1 \\ \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \quad + \dots \text{by Lemma 3.14} \\
 & \quad - \quad \begin{array}{c} \lambda_1 \\ \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \quad + \quad \begin{array}{c} \lambda_1 \\ \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \\
 =_{\lambda} & \quad - \quad \begin{array}{c} \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \quad - \quad \begin{array}{c} \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \\
 & \quad + \quad \begin{array}{c} \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \\
 = & \quad - \quad \begin{array}{c} \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \quad - \quad \begin{array}{c} \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \\
 & \quad + \quad \begin{array}{c} \begin{array}{cccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 0 & 2 & 3 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 2 & 3 \end{array} \end{array} \\
 = & \quad - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\lambda} \vee i) y_{\lambda} - \psi_{\lambda_1} \dots \psi_{n-2} e(\mathbf{i}_{\lambda} \vee i) y_{\lambda} + \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\lambda} \vee i) y_{\lambda} y_n.
 \end{aligned}$$

**Case 4.20d:**  $i = e - 2$  and  $j \neq e - 2$ .

Because we set  $e = 4$  so in this case we have  $i = 2$  and  $j \neq 2$ . Similarly we set  $j = 3$  in this case in order to make the diagrams easier to read. For the other cases with  $j \neq 2$  the argument is similar. By Lemma 3.14,

$$\begin{aligned}
 & e_{\lambda} y_{\lambda} \psi_{\lambda_1} \psi_{\lambda_1+1} \cdots \psi_{n-2} \psi_{n-1} \\
 &= \text{Diagram 1} \\
 &\stackrel{(2.10)}{=} \text{Diagram 2} + \text{Diagram 3} \\
 &=_{\lambda} \text{Diagram 4} = \cdots \\
 &=_{\lambda} \text{Diagram 5} \\
 &\stackrel{(2.10)}{=} \text{Diagram 6} + \text{Diagram 7} \\
 &=_{\lambda} \text{Diagram 8} = \psi_{\lambda_1} \psi_{\lambda_1+1} \cdots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\lambda} \vee i) y_{\lambda}.
 \end{aligned}$$

**Case 4.20e:**  $i = j = e - 2$ .

Because we set  $e = 4$  so in this case we have  $i = j = 2$ . Then by Lemma 3.14,

$$e_{\lambda} y_{\lambda} \psi_{\lambda_1} \psi_{\lambda_1+1} \cdots \psi_{n-2} \psi_{n-1} = \text{Diagram 9}$$



$$\begin{aligned}
 & \stackrel{(2.10)}{=} \text{Diagram 1} + \text{Diagram 2} \\
 & \stackrel{=_{\lambda}}{=} \text{Diagram 3} = \dots \\
 & \stackrel{=_{\lambda}}{=} \text{Diagram 4} \\
 & \stackrel{(2.10)}{=} \text{Diagram 5} + \text{Diagram 6} \\
 & \stackrel{=_{\lambda}}{=} \text{Diagram 7} \\
 & \stackrel{(2.10)}{=} \text{Diagram 8} + \text{Diagram 9} \\
 & = \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} + \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} e(\mathbf{i}_{\tilde{\mu}} \vee i) y_{\tilde{\mu}},
 \end{aligned}$$

which completes the proof.  $\square$

**Remark 4.21.** If  $\lambda_1 > \lambda_2$  and  $\mathbf{t}$  is the last Garnir tableau of  $\lambda$ , by Lemma 4.17 we have

$$\psi_{d(\mathbf{t})} \psi_r = \psi_{a_n, n} \psi_{a_{n-1}, n-1} \dots \psi_{a_{r+2}, r+2} \psi_{a_{r+1}, r+1}.$$

Define  $\mathbf{w}$  to be the last Garnir tableau of shape  $\tilde{\lambda}$ , we see that  $\psi_{d(\mathbf{w})} = \psi_{a_{n-1}, n-1} \dots \psi_{a_{r+2}, r+2} \psi_{a_{r+1}, r+1}$ . Hence  $e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} \psi_{d(\mathbf{w})} \psi_r = \theta_i(\psi_{\mathbf{t} \vee \mathbf{w}} \psi_r)$ .

**Lemma 4.22.** Suppose  $\mathbf{t}$  and  $\tilde{\mathbf{t}}$  are the last Garnir tableau of shape  $\lambda$  and  $\tilde{\lambda}$  respectively with last Garnir entry  $r$ . Set

$$\psi = \begin{cases} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} y_n - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1}, & \text{if } i = e-1, j \neq e-1. \\ \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} y_n - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} - \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \lambda_{n-2}, & \text{if } i = j = e-1. \\ \psi_{\lambda_1} \psi_{\lambda_2} \dots \psi_{n-2} \psi_{n-1}, & \text{otherwise.} \end{cases}$$

For any standard  $\tilde{\lambda}$ -tableau  $\tilde{\mathbf{v}}$ , if  $d(\mathbf{t}) \leq m_{\lambda}$  and  $\tilde{\mathbf{v}} \triangleright \tilde{\mathbf{t}}$ , then

$$\begin{cases} \psi \cdot \theta_i(\psi_{\mathbf{t} \vee \tilde{\mathbf{v}}}) \in R_n^{\geq \mu}, & \text{if } i = j = e-2, \\ \psi \cdot \theta_i(\psi_{\mathbf{t} \vee \tilde{\mathbf{v}}}) \in R_n^{\geq \lambda}, & \text{otherwise.} \end{cases}$$

*Proof.* If it is not the case that  $i = j = e-2$ . By Lemma 4.20 we have

$$\psi \cdot e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} =_{\lambda} e(\mathbf{i}_{\tilde{\lambda}}) y_{\tilde{\lambda}} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1}.$$

Then we have

$$\psi \cdot \theta_i(\psi_{\mathbf{t} \vee \tilde{\mathbf{v}}}) = \psi \cdot e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} \psi_{d(\tilde{\mathbf{v}})} =_{\lambda} e_{\lambda} y_{\lambda} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} \psi_{d(\tilde{\mathbf{v}})},$$

where as  $\tilde{\mathbf{v}} \triangleright \tilde{\mathbf{t}}$ , then  $d(\tilde{\mathbf{v}}) < d(\tilde{\mathbf{t}})$  and

$$l(\psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} \psi_{d(\tilde{\mathbf{v}})}) < l(\psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1}) + l(d(\tilde{\mathbf{t}})) = l(d(\mathbf{t})) \leq m_{\lambda}.$$

Then by Lemma 4.5 we have  $\psi \cdot \theta_i(\psi_{\mathfrak{t}^{\dot{\lambda}}}) \in R_n^{\geq \lambda}$ .

For  $i = j = e - 2$ , set  $\dot{\mu} = (\lambda_1 - 1, \lambda_2 - 1, 1)$ ,  $\gamma = \lambda|_{n-1} = (\lambda_1, \lambda_2 - 1)$  and  $\dot{\gamma} = (\lambda_1 - 1, \lambda_2 - 1)$ . Because  $y_{\dot{\gamma}} = y_{\dot{\mu}}$ . By Lemma 4.20,

$$\begin{aligned} \psi \cdot \theta_i(\psi_{\mathfrak{t}^{\dot{\lambda}}}) &= \psi_{\lambda_1, n} e(\mathbf{i}_{\dot{\lambda}} \vee i) y_{\dot{\lambda}} \psi_{d(\dot{\nu})} =_{\lambda} e_{\lambda} y_{\lambda} \psi_{\lambda_1, n} \psi_{d(\dot{\nu})} - \psi_{\lambda_1, n-1} e(\mathbf{i}_{\dot{\mu}} \vee i) y_{\dot{\mu}} \psi_{d(\dot{\nu})} \\ &= e_{\lambda} y_{\lambda} \psi_{\lambda_1, n} \psi_{d(\dot{\nu})} - \theta_i(\psi_{\lambda_1, n-1} e(\mathbf{i}_{\dot{\gamma}} \vee i) y_{\dot{\mu}} \psi_{d(\dot{\nu})}) \\ &= e_{\lambda} y_{\lambda} \psi_{\lambda_1, n} \psi_{d(\dot{\nu})} - \theta_i(\psi_{\lambda_1, n-1} e(\mathbf{i}_{\dot{\gamma}} \vee i) y_{\dot{\gamma}} \psi_{d(\dot{\nu})}). \end{aligned}$$

Again by Lemma 4.20,  $\psi_{\lambda_1, n-1} e(\mathbf{i}_{\dot{\gamma}} \vee i) y_{\dot{\gamma}} \psi_{d(\dot{\nu})} =_{\gamma} e_{\gamma} y_{\gamma} \psi_{\lambda_1, n-1} \psi_{d(\dot{\nu})}$ . Since  $\gamma = \lambda|_{n-1}$ , by Lemma 3.14,

$$\theta_i(\psi_{\lambda_1, n-1} e(\mathbf{i}_{\dot{\gamma}} \vee i) y_{\dot{\gamma}} \psi_{d(\dot{\nu})}) =_{\lambda} \theta_i(e_{\gamma} y_{\gamma} \psi_{\lambda_1, n-1} \psi_{d(\dot{\nu})}).$$

Therefore

$$\psi \cdot \theta_i(\psi_{\mathfrak{t}^{\dot{\lambda}}}) =_{\lambda} e_{\lambda} y_{\lambda} \psi_{\lambda_1, n} \psi_{d(\dot{\nu})} - \theta_i(e_{\gamma} y_{\gamma} \psi_{\lambda_1, n-1} \psi_{d(\dot{\nu})}).$$

As  $\lambda \in \mathcal{S}_n^{\Lambda}$  and  $|\gamma| = n - 1 < |\lambda|$

$$e_{\gamma} y_{\gamma} \psi_{\lambda_1, n-1} \psi_{d(\dot{\nu})} = \sum_{y \in \text{Shape}(\gamma)} c_{\dot{v}y} \psi_{\dot{v}y} + \sum_{x, y \in \text{Std}(> \gamma)} c_{xy} \psi_{xy}. \quad (4.23)$$

For the first term of the left hand side of (4.23), because  $\gamma = \lambda|_{n-1}$  and  $j = i = e - 2$ , we have  $b_i^{\gamma} = 2$ . By Lemma 3.3 and the definition of  $\gamma$ ,  $\theta_i(\psi_{\dot{v}y}) \in R_n^{\geq \mu}$ . For the second term of the left hand side of (4.23), as  $x, y \in \text{Std}(> \gamma) = \text{Std}(> \lambda|_{n-1})$ ,  $\psi_{xy} \in R_n^{> \lambda|_{n-1}}$ . By Lemma 3.14,  $\theta_i(\psi_{xy}) \in R_n^{> \lambda} \subseteq R_n^{\geq \mu}$ . Therefore,

$$\theta_i(e_{\gamma} y_{\gamma} \psi_{\lambda_1, n-1} \psi_{d(\dot{\nu})}) \in R_n^{\geq \mu}.$$

Finally, as

$$l(\psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} \psi_{d(\dot{\nu})}) < l(\psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1}) + l(d(\dot{\mathfrak{t}})) = l(d(\mathfrak{t})) \leq m_{\lambda},$$

by Lemma 4.5 we have  $e_{\lambda} y_{\lambda} \psi_{\lambda_1, n} \psi_{d(\dot{\nu})} \in R_n^{> \lambda} \subseteq R_n^{\geq \mu}$ . Hence  $\psi \cdot \theta_i(\psi_{\mathfrak{t}^{\dot{\lambda}}}) \in R_n^{\geq \mu}$ . This completes the proof.  $\square$

**Lemma 4.24.** Suppose  $\lambda_1 - \lambda_2 \equiv e - 1 \pmod{e}$ , i.e.  $i = j$ , and  $(\lambda_1 - \lambda_2 + 1)\lambda_2 - 1 \leq m_{\lambda}$ . Let  $\dot{\mathfrak{u}}$  and  $\dot{\mathfrak{v}}$  be standard  $\lambda|_{n-1}$ -tableaux with  $\dot{\mathfrak{u}} \triangleright \mathfrak{t}^{\dot{\lambda}}$ . Assume  $i = j \neq e - 2$ . Then set

$$\psi = \begin{cases} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} y_n - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} - \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2}, & \text{if } i = j = e - 1, \\ \psi_{\lambda_1} \psi_{\lambda_2} \dots \psi_{n-2} \psi_{n-1}, & \text{if } i = j \neq e - 1, e - 2. \end{cases}$$

and we have

$$\psi \cdot \theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) \in R_n^{\geq \lambda}.$$

*Proof.* We assume that  $i = j \neq e - 2$ . First we need to introduce some properties of  $\dot{\mathfrak{u}}$ . Because  $\dot{\mathfrak{u}}$  is a standard  $\lambda|_{n-1}$ -tableau and  $\dot{\mathfrak{u}} \triangleright \mathfrak{t}^{\dot{\lambda}}$ , the only possible choice of  $\dot{\mathfrak{u}}$  is that  $\dot{\mathfrak{u}}|_{n-2} = \mathfrak{t}^{(\lambda_1-1, \lambda_2-1)}$ . Define  $\mathfrak{u}$  and  $\mathfrak{v}$  to be the unique standard  $\lambda$ -tableau with  $\mathfrak{u}|_{n-1} = \dot{\mathfrak{u}}$  and  $\mathfrak{v}|_{n-1} = \dot{\mathfrak{v}}$ , respectively. For example, when  $\lambda = (7, 4)$  and  $e = 3$ , then  $\dot{\mathfrak{u}} = \begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 10 \\ 7 & 8 & 9 \end{smallmatrix}$  and  $\mathfrak{u} = \begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 10 \\ 7 & 8 & 9 & 11 \end{smallmatrix}$ . From the definitions of  $\dot{\mathfrak{u}}$ ,  $\dot{\mathfrak{v}}$  and  $\mathfrak{u}$ ,  $\mathfrak{v}$  we see that  $d(\mathfrak{v}) = d(\dot{\mathfrak{v}})$  and  $l(d(\mathfrak{u})) = l(d(\dot{\mathfrak{u}})) = \lambda_2 - 1$ . Notice that if  $i = j \neq e - 2$ , then  $\mathbf{i}_{\dot{\lambda}} = \mathbf{i}_{\lambda|_{n-1}} \vee i$  and  $y_{\lambda|_{n-1}} = y_{\dot{\lambda}}$ .

Now we consider different cases for  $i, j$ . Suppose  $i = j \neq e - 1, e - 2$ , then

$$\begin{aligned} \psi \cdot \theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) &= \psi_{\lambda_1} \psi_{\lambda_2} \dots \psi_{n-2} \psi_{n-1} \psi_{d(\dot{\mathfrak{u}})} e(\mathbf{i}_{\lambda|_{n-1}} \vee i) y_{\lambda|_{n-1}} \psi_{d(\dot{\mathfrak{v}})} \\ &= \psi_{\lambda_1} \psi_{\lambda_2} \dots \psi_{n-2} \psi_{n-1} \psi_{d(\dot{\mathfrak{u}})} e_{\lambda} y_{\lambda} \psi_{d(\dot{\mathfrak{v}})}. \end{aligned}$$

Recall that  $e \geq 3$ . As  $\lambda_1 - \lambda_2 \equiv e - 1 \pmod{e}$ , we must have  $\lambda_1 - \lambda_2 \geq e - 1 \geq 2$ . Also because of  $\lambda_2 \geq 1$ ,

$$m_{\lambda} \geq (\lambda_1 - \lambda_2 + 1)\lambda_2 - 1 \geq 3\lambda_2 - 1 \geq 2\lambda_2 > l(\psi_{\lambda_1, n} \psi_{d(\dot{\mathfrak{u}})}) = 2\lambda_2 - 1.$$

Hence by Lemma 4.5

$$\begin{aligned} \psi_{\lambda_1} \psi_{\lambda_2} \dots \psi_{n-2} \psi_{n-1} \psi_{d(\dot{\mathfrak{u}})} e_{\lambda} y_{\lambda} \psi_{d(\dot{\mathfrak{v}})} &= \psi_{\lambda_1} \psi_{\lambda_2} \dots \psi_{n-2} \psi_{n-1} \psi_{d(\dot{\mathfrak{u}})} e_{\lambda} y_{\lambda} \psi_{d(\mathfrak{v})} \\ &= \psi_{\lambda_1} \psi_{\lambda_2} \dots \psi_{n-2} \psi_{n-1} \psi_{d(\dot{\mathfrak{u}})} \psi_{\mathfrak{t}^{\dot{\lambda}}\mathfrak{v}} \in R_n^{\geq \lambda}. \end{aligned}$$

Suppose  $i = j = e - 1$ , then

$$\begin{aligned} &\psi \cdot \theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) \\ &= (\psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} y_n - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} - \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2}) \psi_{d(\dot{\mathfrak{u}})} e(\mathbf{i}_{\lambda|_{n-1}} \vee i) y_{\lambda|_{n-1}} \psi_{d(\dot{\mathfrak{v}})} \\ &= \psi_{\lambda_1, n} y_n \psi_{d(\dot{\mathfrak{u}})} \psi_{\mathfrak{t}^{\dot{\lambda}}\mathfrak{v}} - \psi_{\lambda_1+1, n} \psi_{d(\dot{\mathfrak{u}})} \psi_{\mathfrak{t}^{\dot{\lambda}}\mathfrak{v}} - \psi_{\lambda_1, n-1} \psi_{d(\dot{\mathfrak{u}})} \psi_{\mathfrak{t}^{\dot{\lambda}}\mathfrak{v}}. \end{aligned}$$

As  $\psi_{d(\tilde{u})}$  doesn't involve  $\psi_{n-1}$ , by Proposition 4.9 and Lemma 3.12,

$$\psi_{\lambda_1, n} \psi_{d(\tilde{u})} \psi_{t^i \vee} = \psi_{\lambda_1, n} \psi_{d(\tilde{u})} \psi_n \psi_{t^i \vee} \in R_n^{>\lambda}.$$

As  $l(\psi_{\lambda_1+1, n} \psi_{d(\tilde{u})}) = l(\psi_{\lambda_1, n-1} \psi_{d(\tilde{u})}) = \lambda_2 - 1 + \lambda_2 - 1 = 2\lambda_2 - 2 < m_\lambda$ , by Lemma 4.5,  $\psi_{\lambda_1+1, n} \psi_{d(\tilde{u})} \psi_{t^i \vee}$  and  $\psi_{\lambda_1, n-1} \psi_{d(\tilde{u})} \psi_{t^i \vee}$  are both in  $R_n^{\geq \lambda}$ . Hence  $\psi \cdot \theta_i(\psi_{\tilde{u} \vee}) \in R_n^{\geq \lambda}$ .  $\square$

Now we are ready to prove that  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$  when  $\text{Shape}(t)$  has only two rows.

**Proposition 4.25.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$  and  $t$  is the last Garnir tableau of shape  $\lambda$  with  $r$  to be the last Garnir entry and  $l(d(t)) \leq m_\lambda$ , we have

$$\psi_{t^i t} \psi_r = \sum_{(u, v) \triangleright (t^i, t)} c_{uv} \psi_{uv}.$$

*Proof.* By Lemma 4.17, as  $t$  is the last Garnir tableau of shape  $\lambda$ , we have

$$\psi_{d(t)} \psi_r = \psi_{\lambda_1, n} \psi_{\lambda_1-1, n-1} \dots \psi_{\lambda_2, r+1},$$

where  $l(\psi_{\lambda_1, n}) = l(\psi_{\lambda_1-1, n-1}) = \dots = l(\psi_{\lambda_2, r+1}) = \lambda_2$ . We prove the Proposition by induction on  $\lambda_1$ . Recall that we write  $\lambda = (\lambda_1 - 1, \lambda_2)$ ,  $\mu = (\lambda_1, \lambda_2 - 1, 1)$  and  $\dot{\mu} = (\lambda_1 - 1, \lambda_2 - 1, 1)$ .

When  $\lambda_1 = 1$ , by definition of Garnir tableau,  $\lambda_1 = \lambda_2$ . Without loss of generality, we set  $\Lambda = \Lambda_0$ . In this case  $i = 0$  and  $j = e - 1$ . Hence

$$\psi_{t^i t} \psi_r = \psi_{t^i t} \psi_r = e(0, e - 1) \psi_1 = \psi_1 e(e - 1, 0) = 0 \in R_n^{\geq \lambda}.$$

So, when  $\lambda_1 = 1$  the Proposition is true.

As that the Proposition holds for any partition of two rows with the length of its first row less than  $\lambda_1$ , by Lemma 4.20 we have

$$\begin{aligned} \psi_{t^i t} \psi_r &= e_{\lambda} y_{\lambda} \psi_{d(t)} \psi_r = e_{\lambda} y_{\lambda} \psi_{\lambda_1, n} \psi_{d(\tilde{t})} \psi_r \\ &= \begin{cases} \psi_{\lambda_1, n} e(\mathbf{i}_{\tilde{\lambda}} \vee i) y_{\tilde{\lambda}} \psi_{d(\tilde{t})} \psi_r + \psi_{\lambda_1, n-1} e(\mathbf{i}_{\tilde{\mu}} \vee i) y_{\tilde{\mu}} \psi_{d(\tilde{t})} \psi_r \\ \quad = \psi_{\lambda_1, n} \theta_i(\psi_{\tilde{t} \vee} \psi_r) + \psi_{\lambda_1, n-1} \theta_i(e_{\dot{\mu}} y_{\dot{\mu}} \psi_{d(\tilde{t})} \psi_r), & \text{if } i = j = e - 2, \\ \psi \cdot \theta_i(e_{\lambda} y_{\lambda} \psi_{\lambda_1-1, n-1} \dots \psi_{\lambda_2+1, r+2} \psi_{\lambda_2, r+1}) = \psi \cdot \theta_i(\psi_{\tilde{t} \vee} \psi_r), & \text{otherwise.} \end{cases} \end{aligned} \quad (4.26)$$

where  $\tilde{t}$  is the last Garnir tableau with shape  $\tilde{\lambda} = (\lambda_1 - 1, \lambda_2)$ , and

$$\psi = \begin{cases} \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} y_n - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1}, & \text{if } i = e - 1, j \neq e - 1, \\ \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} y_n - \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} - \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2}, & \text{if } i = j = e - 1, \\ \psi_{\lambda_1} \psi_{\lambda_2} \dots \psi_{n-2} \psi_{n-1}, & \text{otherwise.} \end{cases}$$

Now we separate the question into different cases.

**Case 4.25a:**  $i \neq j$ . By (4.26) we have

$$\psi_{t^i t} \psi_r = \psi \cdot \theta_i(\psi_{\tilde{t} \vee} \psi_r).$$

By induction,  $\psi_{\tilde{t} \vee} \psi_r = \sum_{\tilde{v} \triangleright \tilde{t}} c_{\tilde{\lambda} \tilde{v}} \psi_{\tilde{t} \vee} + \sum_{\tilde{u}, \tilde{v} \in \text{Std}(>\tilde{\lambda})} c_{\tilde{u} \tilde{v}} \psi_{\tilde{u} \tilde{v}}$ . Therefore

$$\psi_{t^i t} \psi_r = \sum_{\tilde{v} \in \text{Std}(\tilde{\lambda})} c_{\tilde{\lambda} \tilde{v}} \psi \cdot \theta_i(\psi_{\tilde{t} \vee}) + \sum_{\tilde{u}, \tilde{v} \in \text{Std}(>\tilde{\lambda})} c_{\tilde{u} \tilde{v}} \psi \cdot \theta_i(\psi_{\tilde{u} \tilde{v}}).$$

For  $\tilde{u}, \tilde{v} \in \text{Std}(>\tilde{\lambda})$ , by Lemma 2.36,  $\text{res}(\tilde{u}) = \text{res}(\tilde{t}^i)$ . Because  $i \neq j$ , we always have  $\text{Shape}(\tilde{u}) > \lambda|_{n-1}$ . Hence  $\psi_{\tilde{u} \tilde{v}} \in R_n^{>\lambda|_{n-1}}$ . Therefore by Lemma 3.14 and Lemma 3.12,  $\psi \cdot \theta_i(\psi_{\tilde{u} \tilde{v}}) \in R_n^{>\lambda}$ . So  $\sum_{\tilde{u}, \tilde{v} \in \text{Std}(>\tilde{\lambda})} c_{\tilde{u} \tilde{v}} \psi \cdot \theta_i(\psi_{\tilde{u} \tilde{v}}) \in R_n^{>\lambda}$ .

For  $\tilde{v} \in \text{Std}(\tilde{\lambda})$  with  $\tilde{v} \triangleright \tilde{t}$ , by Lemma 4.22,  $\psi \cdot \theta_i(\psi_{\tilde{t} \vee}) \in R_n^{\geq \lambda}$ . Therefore  $\sum_{\tilde{v} \in \text{Std}(\tilde{\lambda})} c_{\tilde{\lambda} \tilde{v}} \psi \cdot \theta_i(\psi_{\tilde{t} \vee}) \in R_n^{\geq \lambda}$ . These yield  $\psi_{t^i t} \psi_r \in R_n^{\geq \lambda}$ .

**Case 4.25b:**  $i = j \neq e - 2$ . By (4.26) we have

$$\psi_{t^i t} \psi_r = \psi \cdot \theta_i(\psi_{\tilde{t} \vee} \psi_r).$$

By induction,  $\psi_{\tilde{t} \vee} \psi_r = \sum_{\tilde{v} \in \text{Std}(\tilde{\lambda})} c_{\tilde{\lambda} \tilde{v}} \psi_{\tilde{t} \vee} + \sum_{(\tilde{u}, \tilde{v}) \triangleright (t^i, \tilde{t})} c_{\tilde{u} \tilde{v}} \psi_{\tilde{u} \tilde{v}} + \sum_{\tilde{u}, \tilde{v} \in \text{Shape}(>\lambda|_{n-1})} c_{\tilde{u} \tilde{v}} \psi_{\tilde{u} \tilde{v}}$ . Therefore

$$\psi_{t^i t} \psi_r = \sum_{\tilde{v} \in \text{Std}(\tilde{\lambda})} c_{\tilde{\lambda} \tilde{v}} \psi \cdot \theta_i(\psi_{\tilde{t} \vee}) + \sum_{(\tilde{u}, \tilde{v}) \triangleright (t^i, \tilde{t})} c_{\tilde{u} \tilde{v}} \psi \cdot \theta_i(\psi_{\tilde{u} \tilde{v}}) + \sum_{\tilde{u}, \tilde{v} \in \text{Shape}(>\lambda|_{n-1})} c_{\tilde{u} \tilde{v}} \psi \cdot \theta_i(\psi_{\tilde{u} \tilde{v}}).$$

For  $\dot{u}, \dot{v} \in \text{Std}(> \lambda|_{n-1})$ ,  $\psi_{\dot{u}\dot{v}} \in R_n^{>\lambda|_{n-1}}$ . As  $\lambda \in \mathcal{S}_n^\Lambda$ , by Lemma 3.14 we have  $\psi \cdot \theta_i(\psi_{\dot{u}\dot{v}}) \in R_n^{>\lambda}$ . Hence  $\sum_{\dot{u}, \dot{v} \in \text{Std}(>\lambda|_{n-1})} c_{\dot{u}\dot{v}} \psi \cdot \theta_i(\psi_{\dot{u}\dot{v}}) \in R_n^{>\lambda}$ .

For  $\dot{u}, \dot{v} \in \text{Std}(\lambda|_{n-1})$  with  $\dot{u} \triangleright \mathfrak{t}^\lambda$ , because  $m_\lambda \geq d(\mathfrak{t}) = (\lambda_1 - \lambda_2 + 1)\lambda_2 - 1$ , by Lemma 4.24 we have  $\psi \cdot \theta_i(\psi_{\dot{u}\dot{v}}) \in R_n^{\geq \lambda}$ . So  $\sum_{\substack{\dot{u}, \dot{v} \in \text{Std}(\lambda|_{n-1}) \\ (\dot{u}, \dot{v}) \triangleright (\mathfrak{t}^\lambda, \mathfrak{t})}} c_{\dot{u}\dot{v}} \psi \cdot \theta_i(\psi_{\dot{u}\dot{v}}) \in R_n^{\geq \lambda}$ .

For  $\dot{v} \in \text{Std}(\lambda)$  with  $\dot{v} \triangleright \mathfrak{t}$ , by Lemma 4.22,  $\psi \cdot \theta_i(\psi_{\mathfrak{t}\dot{v}}) \in R_n^{\geq \lambda}$ . So  $\sum_{\dot{u}, \dot{v} \in \text{Shape}(>\lambda|_{n-1})} c_{\dot{u}\dot{v}} \psi \cdot \theta_i(\psi_{\dot{u}\dot{v}}) \in R_n^{\geq \lambda}$ . Therefore we have  $\psi_{\mathfrak{t}\mathfrak{t}} \psi_r \in R_n^{\geq \lambda}$ .

**Case 4.25c:**  $i = j = e - 2$ . By (4.26) we have

$$\psi_{\mathfrak{t}\mathfrak{t}} \psi_r = \psi_{\lambda_1, n} \theta_i(\psi_{\mathfrak{t}\mathfrak{t}} \psi_r) + \psi_{\lambda_1, n-1} \theta_i(e_{\mu} y_{\mu} \psi_{d(\mathfrak{t})} \psi_r). \quad (4.27)$$

For the first term of (4.27), by induction,

$$\begin{aligned} \psi_{\lambda_1, n} \theta_i(\psi_{\mathfrak{t}\mathfrak{t}} \psi_r) &= \sum_{\substack{\dot{v} \in \text{Std}(\lambda) \\ \dot{v} \triangleright \mathfrak{t}}} c_{\mathfrak{t}\dot{v}} \psi_{\lambda_1, n} \theta_i(\psi_{\mathfrak{t}\dot{v}}) + \sum_{\substack{\dot{u}, \dot{v} \in \text{Std}(\lambda|_{n-1}) \\ (\dot{u}, \dot{v}) \triangleright (\mathfrak{t}^\lambda, \mathfrak{t})}} c_{\dot{u}\dot{v}} \psi_{\lambda_1, n} \theta_i(\psi_{\dot{u}\dot{v}}) \\ &+ \sum_{\dot{u}, \dot{v} \in \text{Std}(>\lambda|_{n-1})} c_{\dot{u}\dot{v}} \psi_{\lambda_1, n} \theta_i(\psi_{\dot{u}\dot{v}}). \end{aligned} \quad (4.28)$$

For  $\dot{v} \in \text{Std}(\lambda)$  with  $\dot{v} \triangleright \mathfrak{t}$ , by Lemma 4.22, we have  $\psi_{\lambda_1, n} \theta_i(\psi_{\mathfrak{t}\dot{v}}) \in R_n^{\geq \mu}$ . Therefore

$$\sum_{\substack{\dot{v} \in \text{Std}(\lambda) \\ \dot{v} \triangleright \mathfrak{t}}} c_{\mathfrak{t}\dot{v}} \psi_{\lambda_1, n} \theta_i(\psi_{\mathfrak{t}\dot{v}}) \in R_n^{\geq \mu}. \quad (4.29)$$

For  $\dot{u}, \dot{v} \in \text{Std}(\lambda|_{n-1})$  with  $(\dot{u}, \dot{v}) \triangleright (\mathfrak{t}^\lambda, \mathfrak{t})$ , by Lemma 2.36, we have  $\text{res}(\dot{u}) = \mathfrak{i}_\lambda$ . So the choice of  $\dot{u}$  is unique, where  $d(\dot{u}) = \psi_{\lambda_1, n-1}$ . Hence as  $\mathfrak{i}_\mu = \mathfrak{i}_{\lambda|_{n-1}} \vee i$  and  $y_{\lambda|_{n-1}} = y_\mu$

$$\psi_{\lambda_1, n} \theta_i(\psi_{\dot{u}\dot{v}}) = \psi_{\lambda_1, n} \psi_{n-1, \lambda_1} e(\mathfrak{i}_{\lambda|_{n-1}} \vee i) y_{\lambda|_{n-1}} \psi_{d(\dot{v})} = \psi_{\lambda_1, n} \psi_{n-1, \lambda_1} e_{\mu} y_{\mu} \psi_{d(\dot{v})}. \quad (4.30)$$

We work with  $\psi_{\lambda_1, n} \psi_{n-1, \lambda_1} e_{\mu} y_{\mu} = \psi_{\lambda_1} \psi_{\lambda_1+1} \dots \psi_{n-2} \psi_{n-1} \psi_{n-2} \dots \psi_{\lambda_1+1} \psi_{\lambda_1} e_{\mu} y_{\mu}$  first. We define a partition  $\sigma = (\lambda_1, \lambda_2 - 2, 1)$ . Then

$$\begin{aligned} &\psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-2} \psi_{n-1} \psi_{n-2} \psi_{n-3} \dots \psi_{\lambda_1} e_{\mu} y_{\mu} \\ &= \psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-1} \psi_{n-2} \psi_{n-1} \psi_{n-3} \dots \psi_{\lambda_1} e_{\mu} y_{\mu} - \psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-3} \dots \psi_{\lambda_1} e_{\mu} y_{\mu} \\ &= \psi_{n-1} \theta_{i-1}(\psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-2} \psi_{n-3} \dots \psi_{\lambda_1} e_{\sigma} y_{\sigma}) \psi_{n-1} - \psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-3} \dots \psi_{\lambda_1} e_{\mu} y_{\mu}. \end{aligned} \quad (4.31)$$

Consider the lefthand term in (4.31). As  $\lambda \in \mathcal{S}_n^\Lambda$  and  $|\sigma| = n - 1 < |\lambda|$ , we have

$$\psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-2} \psi_{n-3} \dots \psi_{\lambda_1} e_{\sigma} y_{\sigma} = \sum_{u \in \text{Std}(\sigma)} c_{u\mathfrak{t}^\sigma} \psi_{u\mathfrak{t}^\sigma} + \sum_{u, v \in \text{Std}(>\sigma)} c_{uv} \psi_{uv},$$

where  $\text{res}(u) = \mathfrak{i}_{\sigma} \cdot s_{\lambda_1} s_{\lambda_1+1} \dots s_{n-3} s_{n-2} s_{n-3} \dots s_{\lambda_1+1} s_{\lambda_1} = \mathfrak{i}_{\sigma}$  by Lemma 2.36, and  $\text{res}(v) = \mathfrak{i}_{\sigma}$ . Since  $\min\{\lambda_1, \dots, n-2\} = \lambda_1$ , by Lemma 4.10,  $c_{u\mathfrak{t}^\sigma} \neq 0$  implies  $u|_{\lambda_1-1} \triangleright \mathfrak{t}^{\sigma|_{\lambda_1-1}}$ . Then the unique choice for  $u$  is  $u = \mathfrak{t}^\sigma$ . Hence

$$\psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-2} \psi_{n-3} \dots \psi_{\lambda_1} e_{\sigma} y_{\sigma} = c \cdot e_{\sigma} y_{\sigma} + \sum_{u, v \in \text{Std}(>\sigma)} c_{uv} \psi_{uv}.$$

Further more if  $u$  is a standard tableau with  $\text{Shape}(u) > \sigma$  and  $\text{res}(u) = \mathfrak{i}_{\sigma}$ , we must have  $\text{Shape}(u) > \lambda|_{n-1}$ . Hence by Lemma 3.14,

$$\psi_{n-1} \theta_{i-1} \left( \sum_{u, v \in \text{Std}(>\sigma)} c_{uv} \psi_{uv} \right) \psi_{n-1} \in R_n^{>\lambda}.$$

Therefore

$$\begin{aligned} \psi_{n-1} \theta_{i-1}(\psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-2} \psi_{n-3} \dots \psi_{\lambda_1} e_{\sigma} y_{\sigma}) \psi_{n-1} &=_{\lambda} c \cdot \psi_{n-1} \theta_{i-1}(e_{\sigma} y_{\sigma}) \psi_{n-1} \\ &= c \cdot \psi_{n-1}^2 e_{\mu} y_{\mu} = c \cdot (e_{\lambda} y_{\lambda} - e_{\mu} y_{\mu} y_{n-1}). \end{aligned}$$

By Proposition 4.9 we have  $e_{\mu} y_{\mu} y_{n-1} \in R_n^{>\lambda}$ , we have

$$\psi_{n-1} \theta_{i-1}(\psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-2} \psi_{n-3} \dots \psi_{\lambda_1} e_{\sigma} y_{\sigma}) \psi_{n-1} =_{\lambda} c \cdot e_{\lambda} y_{\lambda}. \quad (4.32)$$

For the righthand term in (4.31), as  $\lambda \in \mathcal{S}_n^\Lambda$ ,  $\lambda|_{n-1} \in \mathcal{S}_{n-1}^\Lambda \cap (\mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda)$ . By Lemma 4.11,

$$\psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-3} \dots \psi_{\lambda_1} e_{\lambda|_{n-1}} y_{\lambda|_{n-1}} =_{\lambda|_{n-1}} \sum_{\dot{u} \in \text{Std}(\lambda|_{n-1})} c_{\dot{u}\mathfrak{t}^{\lambda|_{n-1}}} \psi_{\dot{u}\mathfrak{t}^{\lambda|_{n-1}}}.$$

Then by Lemma 3.14,

$$\begin{aligned}
 \psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-2} \dots \psi_{\lambda_1} e_{\mu} y_{\mu} &= \theta_i(\psi_{\lambda_1} \dots \psi_{n-3} \psi_{n-2} \dots \psi_{\lambda_1} e_{\lambda|_{n-1}} y_{\lambda|_{n-1}}) \\
 &=_{\lambda} \sum_{\dot{u} \in \text{Std}(\lambda|_{n-1})} c_{\dot{u} t^{\lambda|_{n-1}}} \theta_i(\psi_{\dot{u} t^{\lambda|_{n-1}}}) \\
 &= \sum_{u \in \text{Std}(\mu)} c_{\dot{u} t^{\lambda|_{n-1}}} \psi_{u t^{\mu}},
 \end{aligned} \tag{4.33}$$

where  $u$  is the unique  $\mu$ -tableau such that  $u|_{n-1} = \dot{u}$ .

So substitute (4.32) and (4.33) to (4.31), we have

$$\psi_{\lambda_1, n} \psi_{n-1, \lambda_1} e_{\mu} y_{\mu} =_{\lambda} \sum_{u \in \text{Std}(\mu)} c_{u t^{\mu}} \psi_{u t^{\mu}} \pm c \cdot \psi_{t^{\lambda} t^{\lambda}}.$$

As  $\dot{v}$  is a standard tableau of shape  $\lambda|_{n-1} = \mu|_{n-1}$ , we can define  $v_1$  and  $v_2$  to be a standard  $\mu$ -tableau and  $\lambda$ -tableau where  $v_1|_{n-1} = v_2|_{n-1} = \dot{v}$ , respectively. Henceforth  $d(v_1) = d(v_2) = d(\dot{v})$  and by (4.30),

$$\begin{aligned}
 \psi_{\lambda_1, n} \theta_i(\psi_{\dot{u} \dot{v}}) &= \psi_{\lambda_1, n} \psi_{n-1, \lambda_1} e_{\mu} y_{\mu} \psi_{d(\dot{v})} =_{\lambda} \sum_{u \in \text{Std}(\mu)} c_{u t^{\mu}} \psi_{u t^{\mu}} \psi_{d(\dot{v})} \pm c \cdot \psi_{t^{\lambda} t^{\lambda}} \psi_{d(\dot{v})} \\
 &= \sum_{u \in \text{Std}(\mu)} c_{u t^{\mu}} \psi_{u t^{\mu}} \psi_{d(v_1)} \pm c \cdot \psi_{t^{\lambda} t^{\lambda}} \psi_{d(v_2)} \\
 &= \sum_{u \in \text{Std}(\mu)} c_{u t^{\mu}} \psi_{u v_1} \pm c \cdot \psi_{t^{\lambda} v_2} \in R_n^{\geq \mu}.
 \end{aligned}$$

Therefore,

$$\sum_{\substack{\dot{u}, \dot{v} \in \text{Std}(\lambda|_{n-1}) \\ (\dot{u}, \dot{v}) > (t^{\lambda}, t^{\lambda})}} c_{\dot{u} \dot{v}} \psi_{\lambda_1, n} \theta_i(\psi_{\dot{u} \dot{v}}) \in R_n^{\geq \mu}. \tag{4.34}$$

Finally, suppose  $\dot{u}, \dot{v} \in \text{Shape}(> \lambda|_{n-1})$ , by Lemma 3.14, we have  $\psi_{\lambda_1, n} \theta_i(\psi_{\dot{u} \dot{v}}) \in R_n^{> \lambda}$ . Therefore

$$\sum_{\dot{u}, \dot{v} \in \text{Std}(> \lambda|_{n-1})} c_{\dot{u} \dot{v}} \psi_{\lambda_1, n} \theta_i(\psi_{\dot{u} \dot{v}}) \in R_n^{> \lambda}. \tag{4.35}$$

Substitute (4.29), (4.34) and (4.35) to (4.28), we have

$$\psi_{\lambda_1, n} \theta_i(\psi_{t^{\lambda} t^{\lambda}} \psi_r) \in R_n^{\geq \mu}. \tag{4.36}$$

For the second term of (4.27), by Lemma 4.20

$$\begin{aligned}
 \psi_{\lambda_1, n-1} \theta_i(e_{\mu} y_{\mu} \psi_{d(\dot{v})} \psi_r) &= \theta_i(\psi_{\lambda_1, n-1} e_{\mu} y_{\mu} \psi_{d(\dot{v})} \psi_r) = \theta_i(e_{\lambda|_{n-1}} y_{\lambda|_{n-1}} \psi_{\lambda_1, n-1} \psi_{d(\dot{v})} \psi_r) \\
 &= e_{\mu} y_{\mu} \psi_{\lambda_1, n-1} \psi_{d(\dot{v})} \psi_r,
 \end{aligned}$$

where by Lemma 4.15, because  $\psi_{\lambda_1, n-1} \psi_{d(\dot{v})} \psi_r$  doesn't involve  $\psi_{n-1}$ ,

$$e_{\mu} y_{\mu} \psi_{\lambda_1, n-1} \psi_{d(\dot{v})} \psi_r \in R_n^{\geq \mu}. \tag{4.37}$$

Therefore substitute (4.36) and (4.37) to (4.27), we have

$$\psi_{t^{\lambda} t^{\lambda}} \psi_r \in R_n^{\geq \mu}.$$

Then by Proposition 2.41 the proof is completed.  $\square$

**Example 4.38.** We give an example of Case 4.25c. Suppose  $\lambda = (7, 4)$ ,  $e = 4$  and  $\Lambda = \Lambda_0$ . Therefore  $i = j = 3$  and

$$t = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 7 & 9 & 10 & 11 \\ \hline 4 & 5 & 6 & 8 & & & \\ \hline \end{array} \quad t^{\lambda} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 8 & 9 & 10 & 11 & & & \\ \hline \end{array},$$

with  $d(t) = s_7 s_8 s_9 s_{10} s_6 s_7 s_8 s_9 s_5 s_6 s_7 s_8 s_4 s_5 s_6$  and  $r = 7$ .

By Lemma 4.20 we have

$$\begin{aligned}
 e_{\lambda} y_{\lambda} \psi_7 \psi_8 \psi_9 \psi_{10} &= e(01230123012) y_4 y_{11} \psi_7 \psi_8 \psi_9 \psi_{10} \\
 &= \psi_7 \psi_8 \psi_9 \psi_{10} e(01230130122) y_4 y_{11} + \psi_7 \psi_8 \psi_9 e(01230130122) y_4 \\
 &= \psi_7 \psi_8 \psi_9 \psi_{10} e(\mathbf{i}_{\lambda} \vee i) y_{\lambda} + \psi_7 \psi_8 \psi_9 e(\mathbf{i}_{\mu} \vee i) y_{\mu} \\
 &= \psi_7 \psi_8 \psi_9 \psi_{10} \theta_i(e(\mathbf{i}_{\lambda}) y_{\lambda}) + \psi_7 \psi_8 \psi_9 \theta_i(e(\mathbf{i}_{\mu}) y_{\mu}),
 \end{aligned}$$

where  $\lambda = (6, 4)$  and  $\mu = (6, 3, 1)$ . Therefore

$$\dot{t} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 7 & 9 & 10 & 11 \\ \hline 4 & 5 & 6 & 8 & & & \\ \hline \end{array}$$

and  $d(\mathbf{i}) = s_6 s_7 s_8 s_9 s_5 s_6 s_7 s_8 s_4 s_5 s_6$ , which indicates

$$\begin{aligned}
\psi_{\mathbf{i}\mathbf{i}}\psi_r &= e_\lambda y_\lambda \psi_7 \psi_8 \psi_9 \psi_{10} \psi_6 \psi_7 \psi_8 \psi_9 \psi_5 \psi_6 \psi_7 \psi_8 \psi_4 \psi_5 \psi_6 \psi_7 \\
&= \psi_7 \psi_8 \psi_9 \psi_{10} \theta_i(e(\mathbf{i}_\lambda) y_\lambda) \psi_6 \psi_7 \psi_8 \psi_9 \psi_5 \psi_6 \psi_7 \psi_8 \psi_4 \psi_5 \psi_6 \psi_7 \\
&\quad + \psi_7 \psi_8 \psi_9 \theta_i(e(\mathbf{i}_\mu) y_\mu) \psi_6 \psi_7 \psi_8 \psi_9 \psi_5 \psi_6 \psi_7 \psi_8 \psi_4 \psi_5 \psi_6 \psi_7 \\
&= \psi_7 \psi_8 \psi_9 \psi_{10} \theta_i(e(\mathbf{i}_\lambda) y_\lambda) \psi_{d(\mathbf{i})} \psi_7 + \psi_7 \psi_8 \psi_9 \theta_i(e(\mathbf{i}_\mu) y_\mu) \psi_{d(\mathbf{i})} \psi_7 \\
&= \psi_{7,11} \theta_i(\psi_{\mathbf{i}\mathbf{i}} \psi_7) + \psi_{7,10} \theta_i(e(\mathbf{i}_\mu) y_\mu \psi_{d(\mathbf{i})} \psi_7).
\end{aligned} \tag{4.39}$$

For the first term of (4.39),

$$\begin{aligned}
\psi_{7,11} \theta_i(\psi_{\mathbf{i}\mathbf{i}} \psi_r) &= \sum_{\substack{\check{\mathbf{v}} \in \text{Std}(\check{\lambda}) \\ \check{\mathbf{v}} \triangleright \check{\mathbf{i}}}} c_{\mathbf{i}\check{\mathbf{v}}} \psi_{7,11} \theta_i(\psi_{\mathbf{i}\check{\mathbf{v}}}) + \sum_{\substack{\check{\mathbf{u}}, \check{\mathbf{v}} \in \text{Std}(\lambda|_{n-1}) \\ (\check{\mathbf{u}}, \check{\mathbf{v}}) \triangleright (\mathbf{i}^\lambda, \mathbf{i})}} c_{\check{\mathbf{u}}\check{\mathbf{v}}} \psi_{7,11} \theta_i(\psi_{\check{\mathbf{u}}\check{\mathbf{v}}}) \\
&\quad + \sum_{\check{\mathbf{u}}, \check{\mathbf{v}} \in \text{Std}(>\lambda|_{n-1})} c_{\check{\mathbf{u}}\check{\mathbf{v}}} \psi_{7,11} \theta_i(\psi_{\check{\mathbf{u}}\check{\mathbf{v}}}).
\end{aligned} \tag{4.40}$$

For  $\check{\mathbf{v}} \in \text{Std}(\check{\lambda})$  with  $\check{\mathbf{v}} \triangleright \check{\mathbf{i}}$ , by Lemma 4.22 we have

$$\sum_{\substack{\check{\mathbf{v}} \in \text{Std}(\check{\lambda}) \\ \check{\mathbf{v}} \triangleright \check{\mathbf{i}}}} c_{\mathbf{i}\check{\mathbf{v}}} \psi_{7,11} \theta_i(\psi_{\mathbf{i}\check{\mathbf{v}}}) \in R_n^{\geq \mu}. \tag{4.41}$$

For  $\check{\mathbf{u}}, \check{\mathbf{v}} \in \text{Std}(\lambda|_{n-1})$  with  $(\check{\mathbf{u}}, \check{\mathbf{v}}) \triangleright (\mathbf{i}^\lambda, \mathbf{i})$ , then  $\text{res}(\check{\mathbf{u}}) = \mathbf{i}_\lambda = 0123013012$ , and because  $\text{Shape}(\check{\mathbf{u}}) = \lambda|_{n-1} = (7, 3)$  with residues

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 0 & 1 & 1 & 2 \\ \hline 3 & 0 & 1 & & & & & \\ \hline \end{array}$$

and

$$\check{\mathbf{u}} \triangleright \mathbf{i}^\lambda = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & & \\ \hline 7 & 8 & 9 & 10 & & & & \\ \hline \end{array}.$$

The only possible choice of  $\check{\mathbf{u}}$  is

$$\check{\mathbf{u}} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 10 & \\ \hline 7 & 8 & 9 & & & & & \\ \hline \end{array},$$

with  $d(\check{\mathbf{u}}) = s_7 s_8 s_9 = \psi_{\lambda_1, n-1}$ . Hence

$$\psi_{7,11} \theta_i(\psi_{\check{\mathbf{u}}\check{\mathbf{v}}}) = \psi_7 \psi_8 \psi_9 \psi_{10} \psi_9 \psi_8 \psi_7 e(0123012301) y_4 \psi_{d(\check{\mathbf{v}})}. \tag{4.42}$$

Notice we have

$$\begin{aligned}
&\psi_7 \psi_8 \psi_9 \psi_{10} \psi_9 \psi_8 \psi_7 e(01230123012) y_4 \\
&= \psi_7 \psi_8 \psi_9 \psi_{10} \psi_9 e(01230130212) \psi_8 \psi_7 y_4 \\
&= \psi_7 \psi_8 \psi_{10} \psi_9 \psi_{10} e(01230130212) \psi_8 \psi_7 y_4 - \psi_7 \psi_8 e(01230130212) \psi_8 \psi_7 y_4 \\
&= \psi_7 \psi_8 \psi_{10} \psi_9 \psi_{10} \psi_8 \psi_7 e(01230123012) y_4 - \psi_7 \psi_8 \psi_8 \psi_7 e(01230123012) y_4 \\
&= \psi_{10} \psi_7 \psi_8 \psi_9 \psi_8 \psi_7 e(01230123021) y_4 \psi_{10} - \psi_7 \psi_8 \psi_8 \psi_7 e(01230123012) y_4 \\
&= \psi_{10} \theta_1(\psi_7 \psi_8 \psi_9 \psi_8 \psi_7 e(0123012302) y_4) \psi_{10} - \psi_7 \psi_8 \psi_8 \psi_7 e(01230123012) y_4 \\
&= \psi_{10} \theta_1(\psi_7 \psi_8 \psi_9 \psi_8 \psi_7 e_\sigma y_\sigma) \psi_{10} - \psi_7 \psi_8 \psi_8 \psi_7 e_\mu y_\mu,
\end{aligned} \tag{4.43}$$

where  $\sigma = (7, 2, 1)$ . Consider the left term of (4.43), because  $|\sigma| < |\lambda|$  and  $\lambda \in \mathcal{S}_n^\Lambda$ , we have

$$\psi_7 \psi_8 \psi_9 \psi_8 \psi_7 e_\sigma y_\sigma = \sum_{\mathbf{u} \in \text{Std}(\sigma)} c_{\mathbf{u}\mathbf{t}^\sigma} \psi_{\mathbf{u}\mathbf{t}^\sigma} + \sum_{\mathbf{u}, \mathbf{v} \in \text{Std}(>\sigma)} c_{\mathbf{u}\mathbf{v}} \psi_{\mathbf{u}\mathbf{v}}.$$

For  $\mathbf{u} \in \text{Std}(\sigma)$ , by Lemma 4.10 and  $\psi_7 \psi_8 \psi_9 \psi_8 \psi_7$  doesn't involve  $\psi_s$  with  $s \leq 6$ , we have  $\mathbf{u}|_6 \triangleright \mathbf{t}^\sigma|_6$ . Then because  $\text{res}(\mathbf{u}) = \mathbf{i}_\sigma \cdot s_7 s_8 s_9 s_8 s_7 = 0123012302$ , by the definition of  $\sigma$

$$[\sigma] = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & & & & & & \\ \hline \end{array} \quad \text{with residues} \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 0 & 1 & 1 & 2 \\ \hline 3 & 0 & & & & & & \\ \hline 2 & & & & & & & \\ \hline \end{array}.$$

Then the only possible choice of  $\mathbf{u}$  is  $\mathbf{t}^\sigma = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \hline 8 & 9 & & & & & & \\ \hline 10 & & & & & & & \\ \hline \end{array}$ . Hence

$$\psi_7 \psi_8 \psi_9 \psi_8 \psi_7 e_\sigma y_\sigma = c \cdot \psi_{\mathbf{t}^\sigma \mathbf{t}^\sigma} + \sum_{\mathbf{u}, \mathbf{v} \in \text{Std}(>\sigma)} c_{\mathbf{u}\mathbf{v}} \psi_{\mathbf{u}\mathbf{v}}.$$

For  $u, v \in \text{Std}(> \sigma)$ , we have  $\text{res}(u) = \mathbf{i}_\sigma = 0123012302$ . It is impossible that  $\text{Shape}(u) = \lambda|_{n-1}$  because  $\mathbf{i}_{\lambda|_{n-1}} = 0123012301$ . Hence  $\sum_{u,v \in \text{Std}(> \sigma)} c_{uv} \psi_{uv} = \sum_{u,v \in \text{Std}(> \lambda|_{n-1})} c_{uv} \psi_{uv} \in R_n^{> \lambda|_{n-1}}$ . So  $\psi_7 \psi_8 \psi_9 \psi_8 \psi_7 e_{\sigma} y_\sigma = c \cdot \psi_{\mathbf{t}^\sigma \mathbf{t}^\sigma} + R_n^{> \lambda|_{n-1}}$  and hence by Lemma 3.14,

$$\begin{aligned} \psi_{10} \theta_1(\psi_7 \psi_8 \psi_9 \psi_8 \psi_7 e_{\sigma} y_\sigma) \psi_{10} &= c \cdot \psi_{10} \theta_1(\psi_{\mathbf{t}^\sigma \mathbf{t}^\sigma}) \psi_{10} + \psi_{10} \theta_1(R_n^{> \lambda|_{n-1}}) \psi_{10} \\ &= c \cdot \psi_{10} e(01230123021) y_4 \psi_{10} + R_n^{> \lambda} \\ &=_{\lambda} c \cdot e(01230123012) y_4 \psi_{10}^2 \\ &= c \cdot e(01230123012) y_4 y_{10} - c \cdot e(01230123012) y_4 y_9 \\ &=_{\lambda} c \cdot e(01230123012) y_4 y_{10} = c \cdot e_{\lambda} y_{\lambda}. \end{aligned} \quad (4.44)$$

For the right term of (4.43), because  $\lambda|_{n-1} \in \mathcal{S}_{n-1}^\Lambda \cap (\mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda)$ , by Lemma 4.11 we have

$$\psi_7 \psi_8 \psi_8 \psi_7 e_{\lambda|_{n-1}} y_{\lambda|_{n-1}} = \sum_{\dot{u} \in \text{Std}(\lambda|_{n-1})} c_{\dot{u} \mathbf{t}^{\lambda|_{n-1}}} \psi_{\dot{u} \mathbf{t}^{\lambda|_{n-1}}} + R_n^{> \lambda|_{n-1}}.$$

Then by Lemma 3.14,

$$\begin{aligned} \psi_7 \psi_8 \psi_8 \psi_7 e_{\mu} y_{\mu} &= \theta_2(\psi_7 \psi_8 \psi_8 \psi_7 e_{\lambda|_{n-1}} y_{\lambda|_{n-1}}) \\ &= \sum_{\dot{u} \in \text{Std}(\lambda|_{n-1})} c_{\dot{u} \mathbf{t}^{\lambda|_{n-1}}} \theta_2(\psi_{\dot{u} \mathbf{t}^{\lambda|_{n-1}}}) + \theta_2(R_n^{> \lambda|_{n-1}}) \\ &= \sum_{u \in \text{Std}(\mu)} c_{\dot{u} \mathbf{t}^{\lambda|_{n-1}}} \psi_{u \mathbf{t}^{\mu}} + R_n^{> \lambda}. \end{aligned} \quad (4.45)$$

Substitute (4.44) and (4.45) back to (4.43), we have

$$\psi_7 \psi_8 \psi_9 \psi_{10} \psi_9 \psi_8 \psi_7 e(01230123012) y_4 = \sum_{u \in \text{Std}(\mu)} c_{u \mathbf{t}^{\mu}} \psi_{u \mathbf{t}^{\mu}} + c \cdot e_{\lambda} y_{\lambda} + R_n^{> \lambda}.$$

Recall  $\dot{v}$  is a standard tableau of shape  $\lambda|_{n-1} = \mu|_{n-1}$ , we can define  $v_1 \in \text{Std}(\mu)$  and  $v_2 \in \text{Std}(\lambda)$  such that  $d(v_1) = d(v_2) = d(\dot{v})$ . Hence by (4.42),

$$\begin{aligned} \psi_{7,11} \theta_i(\psi_{\dot{v}}) &= \psi_7 \psi_8 \psi_9 \psi_{10} \psi_9 \psi_8 \psi_7 e(0123012301) y_4 \psi_{d(\dot{v})} \\ &= \sum_{u \in \text{Std}(\mu)} c_{u v_1} \psi_{u v_1} + c \cdot \psi_{\mathbf{t}^{\lambda} v_2} + R_n^{> \lambda} \in R_n^{\geq \mu}, \end{aligned}$$

which yields

$$\sum_{\substack{\dot{u}, \dot{v} \in \text{Std}(\lambda|_{n-1}) \\ (\dot{u}, \dot{v}) \triangleright (\mathbf{t}^{\lambda}, \mathbf{t})}} c_{\dot{u} \dot{v}} \psi_{7,11} \theta_i(\psi_{\dot{v}}) \in R_n^{\geq \mu}. \quad (4.46)$$

Finally, suppose  $\dot{u}, \dot{v} \in \text{Shape}(> \lambda|_{n-1})$ , by Lemma 3.14 we have  $\psi_{7,11} \theta_i(\psi_{\dot{v}}) \in R_n^{> \lambda}$ . Therefore

$$\sum_{\dot{u}, \dot{v} \in \text{Std}(> \lambda|_{n-1})} c_{\dot{u} \dot{v}} \psi_{7,11} \theta_i(\psi_{\dot{v}}) \in R_n^{> \lambda}. \quad (4.47)$$

Substitute (4.41), (4.46) and (4.47) to (4.40), we have

$$\psi_{7,11} \theta_i(\psi_{\mathbf{t}^{\lambda} \mathbf{t}} \psi_r) \in R_n^{\geq \mu}. \quad (4.48)$$

For the second term of (4.39), by Lemma 4.20

$$\begin{aligned} \psi_{7,10} \theta_i(e(\mathbf{i}_{\mu}) y_{\mu} \psi_{d(\dot{i})} \psi_7) &= \theta_2(\psi_7 \psi_8 \psi_9 e(0123013012) y_4 \psi_{d(\dot{i})} \psi_7) \\ &= \theta_2(e(0123012301) y_4 \psi_7 \psi_8 \psi_9 \psi_{d(\dot{i})} \psi_7) \\ &= e(01230123012) y_4 \psi_7 \psi_8 \psi_9 \psi_{d(\dot{i})} \psi_7 = e_{\mu} y_{\mu} \psi_{7,10} \psi_{d(\dot{i})} \psi_r. \end{aligned}$$

Then by Lemma 4.15, because  $\psi_{7,10} \psi_{d(\dot{i})} \psi_r$  doesn't involve  $\psi_{10}$ , we have  $e_{\mu} y_{\mu} \psi_{7,10} \psi_{d(\dot{i})} \psi_r \in R_n^{\geq \mu}$ . Therefore

$$\psi_{7,10} \theta_i(e(\mathbf{i}_{\mu}) y_{\mu} \psi_{d(\dot{i})} \psi_7) \in R_n^{\geq \mu}. \quad (4.49)$$

Substitute (4.48) and (4.49) to (4.39), we have  $\psi_{\mathbf{t}^{\lambda} \mathbf{t}} \psi_r \in R_n^{\geq \mu}$ . Finally by Proposition 2.41, we have

$$\psi_{\mathbf{t}^{\lambda} \mathbf{t}} \psi_r = \sum_{(u, v) \triangleright (\mathbf{t}^{\lambda}, \mathbf{t})} c_{uv} \psi_{uv}.$$

Finally, we can extend the above Proposition to arbitrary multipartition using arguments similar to those we used in the last section.

**Corollary 4.50.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$  and  $\mathbf{t}$  is the last Garnir tableau of shape  $\lambda$  with  $r$  the last Garnir entry and  $l(d(\mathbf{t})) \leq m_\lambda$ . Therefore, for any standard  $\lambda$ -tableau  $\mathbf{s}$ ,  $\psi_{\mathbf{st}}\psi_r = \sum_{(u,v) \triangleright (\mathbf{s}, \mathbf{t})} c_{uv}\psi_{uv}$ .

*Proof.* Write  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  and  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_k^{(\ell)})$ . If  $\lambda^{(\ell)} = \emptyset$ , then define  $\bar{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)})$ . As  $l(\bar{\lambda}) = \ell - 1 < l(\lambda)$ , we have  $\bar{\lambda} \in \mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$ . By Proposition 3.43,  $\lambda \in \mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$ , and the Corollary follows.

Now suppose  $\lambda^{(\ell)} \neq \emptyset$ . First we assume  $\mathbf{s} = \mathbf{t}^1$ . As  $\mathbf{t}$  is the last Garnir tableau of shape  $\lambda$ ,  $k \geq 2$ . Setting  $m = \lambda_{k-1}^{(\ell)} + \lambda_k^{(\ell)}$ . As  $\mathbf{t}$  is the last Garnir tableau, by the definition we see that  $\mathbf{t}|_{n-m} = \mathbf{t}^1|_{n-m}$  and  $k \geq 2$ . Define  $i$  to be the residue of the node  $(k-1, 1, \ell)$ ,  $\Lambda' = \Lambda_i$ , and  $\tilde{\mathbf{t}}$  to be the last Garnir tableau of shape  $(\lambda_{k-1}^{(\ell)}, \lambda_k^{(\ell)})$ . If we write  $\mu = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)}, \mu^{(\ell)})$  with  $\mu^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_{k-2}^{(\ell)})$  and  $\gamma = (\lambda_{k-1}^{(\ell)}, \lambda_k^{(\ell)})$ , then

$$\psi_{\mathbf{t}^1\mathbf{t}}\psi_r = \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\tilde{\mathbf{t}}\tilde{\mathbf{t}}}\hat{\psi}_{r-(n-m)})y_\mu.$$

Recall that  $\hat{\psi}_{\tilde{\mathbf{t}}\tilde{\mathbf{t}}}$  and  $\hat{\psi}_{r-(n-m)}$  are elements of  $\mathcal{R}_m$  and  $\psi_{\tilde{\mathbf{t}}\tilde{\mathbf{t}}}$  and  $\psi_{r-(n-m)}$  are elements of  $\mathcal{R}_m^{\Lambda'}$ . Then by Proposition 4.25, we have  $\psi_{\tilde{\mathbf{t}}\tilde{\mathbf{t}}}\psi_{r-(n-m)} \in R_n^{\geq \gamma}$ . Therefore we can write  $\psi_{\tilde{\mathbf{t}}\tilde{\mathbf{t}}}\psi_{r-(n-m)} = \sum_{u,v \in \text{Std}(\gamma)} c_{uv}\psi_{uv} + \sum_{u,v \in \text{Std}(>\gamma)} c_{uv}\psi_{uv}$  and hence  $\hat{\psi}_{\tilde{\mathbf{t}}\tilde{\mathbf{t}}}\hat{\psi}_{r-(n-m)} = \sum_{u,v \in \text{Std}(\gamma)} c_{uv}\hat{\psi}_{uv} + \sum_{u,v \in \text{Std}(>\gamma)} c_{uv}\hat{\psi}_{uv} + r$  where  $r \in N_m^{\Lambda'}$ . Therefore

$$\psi_{\mathbf{t}^1\mathbf{t}}\psi_r = \sum_{u,v \in \text{Std}(\gamma)} c_{uv}\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{uv})y_\mu + \sum_{u,v \in \text{Std}(>\gamma)} c_{uv}\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{uv})y_\mu + \hat{\theta}_{\mathbf{i}_\mu}(r)y_\mu.$$

For  $u, v \in \text{Std}(\gamma)$ , by Corollary 3.37 we have  $\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{uv}) \in R_n^{\geq \lambda}$ . Hence  $\sum_{u,v \in \text{Std}(\gamma)} c_{uv}\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{uv})y_\mu \in R_n^{\geq \lambda}$ .

For  $u, v \in \text{Std}(>\gamma)$ , write  $\text{Shape}(u) = \text{Shape}(v) = \sigma$  and  $\nu = \mu \vee \sigma$ . By Corollary 3.35 we have  $\nu > \lambda = \mu \vee \gamma$ . Then by Corollary 3.44 and Lemma 3.14,  $\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{uv}) \in R_n^{\geq \lambda}$ . Hence  $\sum_{u,v \in \text{Std}(>\gamma)} c_{uv}\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{uv})y_\mu \in R_n^{\geq \lambda}$ .

Finally by Lemma 3.33,  $\hat{\theta}_{\mathbf{i}_\mu}(r)y_\mu \in R_n^{\geq \lambda}$ . These yield that

$$\psi_{\mathbf{t}^1\mathbf{t}}\psi_r = \hat{\theta}_{\mathbf{i}_\mu}(\psi_{\tilde{\mathbf{t}}\tilde{\mathbf{t}}}\psi_{r-(n-m)})y_\mu \in R_n^{\geq \lambda}.$$

Now choose any  $\mathbf{s} \in \text{Std}(\lambda)$ . Because  $\psi_{\mathbf{t}^1\mathbf{t}}\psi_r \in R_n^{\geq \lambda}$ , we have

$$\psi_{\mathbf{t}^1\mathbf{t}}\psi_r = \lambda \sum_{v \in \text{Std}(\lambda)} c_{\mathbf{t}^1\mathbf{v}}\psi_{\mathbf{t}^1\mathbf{v}}.$$

Hence

$$\psi_{\mathbf{st}}\psi_r = \psi_{d(\mathbf{s})}^*\psi_{\mathbf{t}^1\mathbf{t}}\psi_r = \lambda \sum_{v \in \text{Std}(\lambda)} c_{\mathbf{t}^1\mathbf{v}}\psi_{d(\mathbf{s})}^*\psi_{\mathbf{t}^1\mathbf{v}} = \sum_{v \in \text{Std}(\lambda)} c_{\mathbf{t}^1\mathbf{v}}\psi_{\mathbf{sv}}.$$

Therefore,  $\psi_{\mathbf{st}}\psi_r \in R_n^{\geq \lambda}$ . By Proposition 2.41 we completes the proof.  $\square$

**Remark 4.51.** Generally it is not easy to find  $c_{uv}$ . Kleshchev-Mathas-Ram [14] explicitly describes how to compute  $c_{uv}$  where  $\text{Shape}(u) = \text{Shape}(v) = \lambda$ . This paper also gives an implicit method to compute these coefficients.

#### 4.5. Completion of the $\psi$ -problem

In this subsection we are going to prove that  $\psi_{\mathbf{st}}\psi_r \in R_n^{\geq \lambda}$ . Corollary 4.7 shows that if  $\mathbf{t} \cdot s_r$  is standard and  $d(\mathbf{t}) \cdot s_r$  is reduced then our claim is true. It remains to consider the case when  $\mathbf{t} \cdot s_r$  is not standard or  $d(\mathbf{t}) \cdot s_r$  is not reduced.

First we introduce two special conditions on  $\mathbf{t} \in \text{Std}(\lambda)$  and  $r$  for  $1 \leq r \leq n-1$ .

**Definition 4.52.** Suppose  $\mathbf{t}$  is a standard  $\lambda$ -tableau and  $1 \leq r \leq n-1$ .

(a). If there exists a reduced expression  $s_{r_1}s_{r_2}\dots s_{r_{l-1}}s_{r_l}$  of  $d(\mathbf{t})$  such that  $|r - r_l| > 1$ , then  $\mathbf{t}$  is **unlocked by  $s_r$  in type I**.

(b). If there exists a reduced expression  $s_{r_1}s_{r_2}\dots s_{r_{l-1}}s_{r_l}$  of  $d(\mathbf{t})$  such that  $r_{l-1} = r$  and  $r = r_l \pm 1$ , then  $\mathbf{t}$  is **unlocked by  $s_r$  in type II**.

The following Lemmas show that if  $l(\mathbf{t}) \leq m_\lambda$  and  $\mathbf{t}$  is unlocked by  $s_r$  in type I or type II, then we have  $\psi_{\mathbf{st}}\psi_r \in R_n^{\geq \lambda}$ .

**Lemma 4.53.** Suppose  $\mathbf{t}$  and  $\mathbf{s}$  are two standard  $\lambda$ -tableaux with  $d(\mathbf{t}) = d(\mathbf{s}) \cdot s_k$  for some  $k$  and  $l(d(\mathbf{t})) = l(d(\mathbf{s})) + 1$ . If for some  $r \notin \{k-1, k, k+1\}$ ,  $\mathbf{t} \cdot s_r$  is not standard or  $d(\mathbf{t}) \cdot s_r$  is not reduced, then  $\mathbf{s} \cdot s_r$  is not standard or  $d(\mathbf{s}) \cdot s_r$  is not standard, respectively.

*Proof.* When  $\mathbf{t} \cdot s_r$  is not standard, then  $r$  and  $r+1$  in  $\mathbf{t}$  are adjacent, either in the same row or in the same column. Since  $d(\mathbf{t}) = d(\mathbf{s}) \cdot s_k$ , we have  $\mathbf{s} = \mathbf{t} \cdot s_k$ . As  $r \notin \{k-1, k, k+1\}$ ,  $r$  and  $r+1$  are in the same nodes in  $\mathbf{t}$  as in  $\mathbf{s}$ . Hence  $\mathbf{s} \cdot s_r$  is not standard as well.

When  $d(\mathbf{t}) \cdot s_r$  is not reduced,  $d(\mathbf{t})(r) > d(\mathbf{t})(r+1)$ . As  $d(\mathbf{t}) = d(\mathbf{s}) \cdot s_k$  we have  $d(\mathbf{t})(r) = d(\mathbf{s}) \cdot s_k(r) = d(\mathbf{s})(r)$  and  $d(\mathbf{t})(r+1) = d(\mathbf{s}) \cdot s_k(r+1) = d(\mathbf{s})(r+1)$  as  $r \notin \{k-1, k, k+1\}$ . Hence  $d(\mathbf{s})(r) > d(\mathbf{s})(r+1)$ . Therefore,  $d(\mathbf{s}) \cdot s_r$  is not reduced. This completes the proof.  $\square$



**Example 4.54.** Suppose  $t = \begin{smallmatrix} 1 & 2 & 7 & 9 \\ 3 & 5 & 8 \\ 4 & 6 & 10 \end{smallmatrix}$  and  $s = \begin{smallmatrix} 1 & 2 & 7 & 8 \\ 3 & 4 & 9 \\ 5 & 6 & 10 \end{smallmatrix}$ . Then

$$\begin{aligned} d(t) &= s_4 s_5 s_6 s_7 s_8 s_6 s_7 s_3 s_4 s_5 s_6 s_4, \\ d(s) &= s_7 s_8 s_4 s_5 s_6 s_7 s_3 s_4 s_5 s_6 s_4. \end{aligned}$$

Set  $k = 8$ , we have  $d(t) = d(s) \cdot s_8$  and  $l(d(t)) = l(d(s)) + 1$ . Let  $r = 3$ , i.e.  $r \notin \{k-1, k, k+1\}$  and  $t \cdot s_r$  is not standard. We see that  $s \cdot s_r$  is not standard either. Similarly, let  $r = 6$ , i.e.  $r \notin \{k-1, k, k+1\}$  and  $d(t) \cdot s_r$  is not reduced. Hence  $d(s) \cdot s_r$  is not reduced either.

**Lemma 4.55.** Suppose  $\lambda \in \mathcal{J}_n^\Lambda$  and  $t$  is a standard  $\lambda$ -tableau with  $l(d(t)) \leq m_\lambda$ . If  $t$  is unlocked by  $s_r$  in type I, then  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$  for any standard  $\lambda$ -tableau  $s$ .

*Proof.* Suppose  $t \cdot s_r$  is standard and  $d(t) \cdot s_r$  is reduced, by Corollary 4.7,  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$ .

Suppose  $t \cdot s_r$  is not standard or  $d(t) \cdot s_r$  is not reduced. Since  $t$  is a standard  $\lambda$ -tableau unlocked by  $s_r$  in type I, by Definition 4.52, there exists a reduced expression  $s_{r_1} s_{r_2} \dots s_{r_l}$  of  $d(t)$  such that  $|r - r_l| > 1$ . Define  $w = t^{\lambda} \cdot s_{r_1} s_{r_2} \dots s_{r_{l-1}}$ . By Lemma 2.31,  $w$  is a standard  $\lambda$ -tableau. It is easy to see that  $d(t) = d(w) \cdot s_{r_l}$  and  $l(d(t)) = l(d(w)) + 1$ . Hence by Lemma 4.53,  $w \cdot s_r$  is not standard or  $d(w) \cdot s_r$  is not reduced. So  $\psi_{sw} \psi_r = \sum_{\substack{v \in \text{Std}(\lambda) \\ v \triangleright w}} c_{sv} \psi_{sv}$  because  $l(d(w)) = l(d(t)) - 1 < w_\lambda$ .

We can write  $d(t) = d(w) \cdot s_{r_l}$  as a reduced expression. By Lemma 4.6, we have

$$\sum_{\substack{v \in \text{Std}(\lambda) \\ v \triangleright t}} c_{sv} \psi_{sv} \psi_r = \psi_{st} \psi_r - \psi_{d(s)}^* e_{\lambda} y_{\lambda} \psi_{d(w)} \psi_{r_l} \psi_r = \psi_{st} \psi_r - \psi_{sw} \psi_r \psi_{r_l}.$$

Because  $v \triangleright t$  and  $l(d(v)) < l(d(t)) \leq m_\lambda$ , we have  $\sum_{\substack{v \in \text{Std}(\lambda) \\ v \triangleright t}} c_{sv} \psi_{sv} \psi_r \in R_n^{\geq \lambda}$ . We can write  $\psi_{sw} \psi_r \psi_{r_l} = \sum_{\substack{v \in \text{Std}(\lambda) \\ v \triangleright w}} c_{sv} \psi_{sv} \psi_{r_l}$ . Because  $v \triangleright w$ , we have  $l(d(v)) < l(d(w)) < m_\lambda$ , which yields that  $\psi_{sw} \psi_r \psi_{r_l} \in R_n^{\geq \lambda}$ . Therefore we have  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$ .  $\square$

**Lemma 4.56.** Suppose  $t$  is a standard  $\lambda$ -tableau and that there exists a standard  $\lambda$ -tableau  $w$  such that  $d(t) = d(w) \cdot s_r s_{r+1}$  for some  $r$  and  $l(d(t)) = l(d(w)) + 2$ . If  $t \cdot s_r$  is not standard or  $d(t) \cdot s_r$  is not reduced, then  $w \cdot s_{r+1}$  is not standard or  $d(w) \cdot s_{r+1}$  is not reduced, respectively.

Similarly suppose  $d(t) = d(w) \cdot s_r s_{r-1}$  for some  $r$  and  $l(d(t)) = l(d(w)) + 2$ . If  $t \cdot s_r$  is not standard or  $d(t) \cdot s_r$  is not reduced, then  $w \cdot s_{r-1}$  is not standard or  $d(w) \cdot s_{r-1}$  is not reduced, respectively.

*Proof.* Suppose  $d(t) = d(w) \cdot s_r s_{r+1}$ . If  $t \cdot s_r$  is not standard,  $r$  and  $r+1$  are adjacent in  $t$ . But  $r$  and  $r+1$  occupy the same positions as  $r+1$  and  $r+2$ , respectively in  $w$ . So  $w \cdot s_{r+1}$  is not standard. If  $d(t) \cdot s_r$  is not reduced, as  $d(w)^{-1}(r+1) = d(t)^{-1}(r)$  and  $d(w)^{-1}(r+2) = d(t)^{-1}(r+1)$ , by Proposition 2.27, then  $d(t) \cdot s_r$  is not reduced implies  $d(w) \cdot s_{r+1}$  is not reduced. The other case is similar.

**Remark 4.57.** In Lemma 4.53 and Lemma 4.56, when we say  $d(t) = d(s) \cdot s_r$  or  $d(t) = d(s) \cdot s_r s_{r+1}$ , it means  $d(t)$  and  $d(s) \cdot s_r$  or  $d(t)$  and  $d(s) \cdot s_r s_{r+1}$  are the same as permutations.

**Example 4.58.** Let  $t = \begin{smallmatrix} 1 & 2 & 3 & 12 \\ 4 & 5 & 6 & 13 \\ 7 & 8 & 11 \\ 9 & 10 & 14 \end{smallmatrix}$ . Suppose  $s = \begin{smallmatrix} 1 & 2 & 3 & 12 \\ 4 & 5 & 6 & 13 \\ 7 & 8 & 9 \\ 10 & 11 & 14 \end{smallmatrix}$ , we have

$$\begin{aligned} d(t) &= s_8 s_9 s_{10} s_{11} s_{12} s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_9 s_{10}, \\ d(s) &= s_8 s_9 s_{10} s_{11} s_{12} s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11}. \end{aligned}$$

So we have  $d(t) = d(s) s_9 s_{10}$  and therefore  $r = 9$ . We see that  $t \cdot s_r$  and  $s \cdot s_{r+1}$  are both non-standard.

Suppose  $s = \begin{smallmatrix} 1 & 2 & 3 & 11 \\ 4 & 5 & 6 & 13 \\ 7 & 8 & 10 \\ 9 & 12 & 14 \end{smallmatrix}$ , we have

$$\begin{aligned} d(t) &= s_8 s_9 s_{10} s_{11} s_{12} s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_9 s_{10}, \\ d(s) &= s_8 s_9 s_{10} s_{11} s_{12} s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_9. \end{aligned}$$

So we have  $d(t) = d(s) s_{11} s_{10}$  and therefore  $r = 11$ . We see that  $d(t) \cdot s_r$  and  $d(s) \cdot s_{r-1}$  are both non-reduced because in  $t$ ,  $r$  is below  $r+1$  and in  $s$ ,  $r-1$  is below  $r$ .

**Lemma 4.59.** Suppose  $\lambda \in \mathcal{J}_n^\Lambda$  and  $t$  is a standard  $\lambda$ -tableau with  $l(d(t)) \leq m_\lambda$ . If  $t$  is unlocked by  $s_r$  in type II, then  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$  for any standard  $\lambda$ -tableau  $s$ .

*Proof.* Suppose  $t \cdot s_r$  is standard and  $d(t) \cdot s_r$  is reduced. Then, by Corollary 4.7,  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$ .

Suppose  $t \cdot s_r$  is not standard or  $d(t) \cdot s_r$  is not reduced. Then, by Definition 4.52, there exists a reduced expression  $s_{r_1} s_{r_2} \dots s_{r_{l-1}} s_{r_l}$  for  $d(t)$  such that  $r_{l-1} = r$  and  $r = r_l \pm 1$ . Without loss of generality, set  $r = r_l - 1$ . Define  $w = t^l s_{r_1} s_{r_2} \dots s_{r_{l-2}}$ . By Lemma 2.31,  $w$  is a standard  $\lambda$ -tableau. It is easy to see that  $d(t) = d(w) \cdot s_r s_{r+1}$  and  $l(d(t)) = l(d(w)) + 2$ . Hence, by Lemma 4.56,  $w \cdot s_{r+1}$  is not standard or  $d(w) \cdot s_{r+1}$  is not reduced. So  $\psi_{sw} \psi_{r+1} =_\lambda \sum_{v \in \text{Std}(\lambda)} c_{sv} \psi_{sv}$  because  $l(d(w)) = l(d(t)) - 2 < m_\lambda$ .

Because  $d(t) = d(w) \cdot s_r s_{r+1}$  as a reduced expression, by Lemma 4.6, we have

$$\sum_{\substack{v \in \text{Std}(\lambda) \\ v \triangleright t}} c_{sv} \psi_{sv} \psi_r =_\lambda \psi_{st} \psi_r - \psi_{d(s)}^* e_{\lambda} y_{\lambda} \psi_{d(w)} \psi_r \psi_{r+1} \psi_r = \psi_{st} \psi_r - \psi_{sw} \psi_r \psi_{r+1} \psi_r. \quad (4.60)$$

Now  $v \triangleright t$ , so  $l(d(v)) < l(d(t)) \leq m_\lambda$ . Hence we have

$$\sum_{\substack{v \in \text{Std}(\lambda) \\ v \triangleright t}} c_{sv} \psi_{sv} \psi_r \in R_n^{\geq \lambda}. \quad (4.61)$$

Let  $\text{res}(w) = i_1 i_2 \dots i_n$  be the residue sequence of  $w$ . Then

$$\psi_{sw} \psi_r \psi_{r+1} \psi_r = \begin{cases} \psi_{sw} \psi_{r+1} \psi_r \psi_{r+1} \pm \psi_{sw}, & \text{if } i_r = i_{r+2} = i_{r+1} \pm 1, \\ \psi_{sw} \psi_{r+1} \psi_r \psi_{r+1}, & \text{otherwise.} \end{cases}$$

Because  $\psi_{sw} \psi_{r+1} =_\lambda \sum_{v \in \text{Std}(\lambda)} c_{sv} \psi_{sv}$ ,

$$\psi_{sw} \psi_{r+1} \psi_r \psi_{r+1} =_\lambda \sum_{v \triangleright w} c_{sv} \psi_{sv} \psi_r \psi_{r+1}.$$

Since  $v \triangleright w$ ,  $l(d(v)) < l(d(w)) = l(d(t)) - 2 \leq m_\lambda - 2$ . Hence,  $l(\psi_{d(v)} \psi_r \psi_{r+1}) = l(d(v)) + 2 < m_\lambda$ . By Lemma 4.5 we have  $\psi_{sv} \psi_r \psi_{r+1} \in R_n^{\geq \lambda}$  if  $v \triangleright w$ . Therefore, we always have

$$\psi_{sw} \psi_{r+1} \psi_r \psi_{r+1} \in R_n^{\geq \lambda} \quad (4.62)$$

in both cases. Substituting (4.61) and (4.62) into (4.60) shows that  $\psi_{st} \psi_r \in R_n^{\geq \lambda}$ .  $\square$

The following Lemmas are technical results which we will use later.

**Lemma 4.63.** Suppose  $t \in \text{Std}(\lambda)$  with  $d(t) = s_{n-1} s_{n-2} \dots s_{r+1}$ , and  $t \cdot s_r$  is not standard. Then  $t$  is the last Garnir tableau of shape  $\lambda$ .

*Proof.* As  $d(t)$  is the standard expression, we have  $w_n = s_{n-1}$ ,  $w_{n-1} = s_{n-2}, \dots, w_{r+2} = s_{r+1}$  and  $t = t^{(1)} = t^{(2)} = \dots = t^{(r+1)}$ . Write  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  and  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_k^{(\ell)})$ , as  $t^{(n)} = t^l \cdot w_n = t^l \cdot s_{n-1}$  is standard,  $n-1$  and  $n$  are not adjacent in  $t^l$ . This forces  $\lambda_k^{(\ell)} = 1$ .

By (2.29) we have

$$t^{-1}(k) = \begin{cases} (t^l)^{-1}(k-1), & \text{if } r+2 \leq k \leq n, \\ (t^l)^{-1}(n), & \text{if } k = r+1, \\ (t^l)^{-1}(k), & \text{otherwise.} \end{cases}$$

and since  $t \cdot s_r$  is not standard,  $r$  and  $r+1$  are adjacent in  $t$ . As  $t^{-1}(r+1) = (t^l)^{-1}(n) = (k, \lambda_k^{(\ell)}, \ell) = (k, 1, \ell)$ , we must have  $t^{-1}(r) = (k-1, 1, \ell)$ . This shows that  $t$  is the last Garnir tableau with shape  $\lambda$ .  $\square$

**Example 4.64.** Suppose  $\lambda = (4, 4, 1)$  and  $t = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 7 & 8 & 9 \\ \hline 6 & & & \end{array}$ . Therefore  $d(t) = s_8 s_7 s_6$  and  $t \cdot s_5$  is not standard. Notice that  $t$  is the last Garnir tableau of shape  $\lambda$ .

**Lemma 4.65.** Suppose  $t \in \text{Std}(\lambda)$  with  $d(t) = s_r s_{r+1} \dots s_{n-2}$ , and  $t \cdot s_{n-1}$  is not standard. Then  $t$  is the last Garnir tableau of shape  $\lambda$ .

*Proof.* As  $d(t)$  is the standard expression, we have  $w_n = s_r s_{r+1} \dots s_{n-2}$  and  $w_{n-1} = \dots w_1 = 1$ . Write  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  and  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_k^{(\ell)})$ , as  $t = t^l \cdot w_n$ , we have  $t^{-1}(n) = (k, \lambda_k^{(\ell)}, \ell)$ . By (2.29) we have

$$t^{-1}(k) = \begin{cases} (t^l)^{-1}(k+1), & \text{if } r \leq k \leq n-2, \\ (t^l)^{-1}(r), & \text{if } k = n-1, \\ (t^l)^{-1}(k), & \text{otherwise.} \end{cases}$$

As  $t \cdot s_{n-1}$  is not standard,  $n-1$  and  $n$  are adjacent in  $t$ . So in  $r$  and  $n$  are adjacent in  $t^l$ . But  $r \leq n-2$ . Hence  $r$  has to be on the above of  $n$  in  $t^l$ . i.e.  $(t^l)^{-1}(r) = t^{-1}(n-1) = (k-1, \lambda_k^{(\ell)}, \ell)$ . This shows that  $\lambda_{k-1}^{(\ell)} = \lambda_k^{(\ell)}$  and  $t$  is the last Garnir tableau of shape  $\lambda$ .  $\square$

**Example 4.66.** Suppose  $\lambda = (4, 3, 3)$  and  $t = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 9 & \\ \hline 8 & 9 & 10 & \\ \hline \end{array}$ . Therefore  $d(t) = s_7 s_8$  and  $t \cdot s_9$  is not standard. Notice that  $t$  is the last Garnir tableau of shape  $\lambda$ .

Recall that we have a standard expression for  $d(t)$  such that  $d(t) = w_n w_{n-1} \dots w_1$ , where  $w_i = 1$  or  $w_i = s_{a_i} s_{a_i+1} \dots s_{i-1}$  for  $1 \leq a_i \leq i-1$  and  $1 \leq i \leq n$ .

**Lemma 4.67.** Suppose  $t \in \text{Std}(\lambda)$  and  $d(t) = w_n w_{n-1} \dots w_1$  is the standard expression where  $w_i \neq 1$  if  $i \geq r+2$  or  $i = r$  and  $w_i = 1$  if  $i < r$  or  $i = r+1$ , i.e.  $d(t) = w_n w_{n-1} \dots w_{r+2} w_r$ . If  $t \cdot s_r$  is not standard, then  $l(w_i) \geq l(w_r) + 1$  for  $i \geq r+2$ .

*Proof.* Write  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  and  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_k^{(\ell)})$ . Because  $t \cdot s_r$  is not standard,  $r$  and  $r+1$  are adjacent in  $t$ . By (2.29), as  $w_i \neq 1$  for  $i \geq r+2$

$$(k, \lambda_k^{(\ell)}, \ell) = (t^{(n+1)})^{-1}(n) = (t^{(n)})^{-1}(n-1) = \dots = (t^{(r+3)})^{-1}(r+2) = (t^{(r+2)})^{-1}(r+1).$$

Notice that  $w_{r+1} = 1$  and  $w_r$  doesn't involve  $s_r$  or  $s_{r+1}$ , we have

$$(k, \lambda_k^{(\ell)}, \ell) = (t^{(r+2)})^{-1}(r+1) = (t^{(r+1)})^{-1}(r+1) = (t^{(r)})^{-1}(r+1).$$

Since  $w_r \neq 1$  and  $w_{r+1} = 1$ , recall  $w_r = s_{a_r} s_{a_r+1} \dots s_{r-2} s_{r-1}$ , by (2.29) we have

$$(t^{(r+2)})^{-1}(a_r) = (t^{(r+1)})^{-1}(a_r) = (t^{(r)})^{-1}(r).$$

Since  $w_i = 1$  for  $i < r$ , we have  $t^{(r)} = t$ . Then  $t^{-1}(r+1) = (k, \lambda_k^{(\ell)}, \ell)$ . Because  $a_r \leq r-1 < r+1$ , by (2.29),  $a_r$  is not on the left of  $r+1$  in  $t^{(r+2)}$  because  $t^{(r+2)}|_{r+1} = t^\mu$  with  $\mu = \text{Shape}(t^{(r+2)}|_{r+1})$ . As  $r$  and  $r+1$  are adjacent in  $t$  and  $(t^{(r+2)})^{-1}(a_r) = (t^{(r)})^{-1}(r) = t^{-1}(r)$ , we must have  $t^{-1}(r) = (k-1, \lambda_{k-1}^{(\ell)}, \ell)$ . Therefore by the definition of the standard expression, we have  $l(w_r) = \lambda_k^{(\ell)} - 1$ .

Since  $(k, \lambda_k^{(\ell)}, \ell) = (t^{(n+1)})^{-1}(n) = (t^{(n)})^{-1}(n-1) = \dots = (t^{(r+2)})^{-1}(r+1)$  and (2.29), we have  $l(w_i) \geq \lambda_k^{(\ell)} = l(w_r) + 1$  for all  $i \geq r+2$ .  $\square$

**Lemma 4.68.** Suppose  $t \in \text{Std}(\lambda)$  and  $d(t) = w_n w_{n-1} \dots w_1$  with  $w_i \neq 1$  if  $i > r+2$  or  $i = r$  and  $w_i = 1$  if  $i < r$  or  $i = r+1$ . If  $l(w_i) = l(w_r) + 1$  for all  $i \geq r+2$ , i.e.  $d(t) = w_n w_{n-1} \dots w_{r+2} w_r$ , and  $t \cdot s_r$  is not standard, then  $t$  is the last Garnir tableau of shape  $\lambda$ .

*Proof.* Write  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$  and  $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_k^{(\ell)})$ . From the proof of Lemma 4.67 we have seen that  $l(w_i) = \lambda_k^{(\ell)}$  for  $i \geq r+2$  and  $l(w_r) = \lambda_k^{(\ell)} - 1$ . Therefore if  $t^1(k-1, \lambda_{k-1}^{(\ell)}, \ell) = t$ ,

$$\begin{cases} w_n = s_t s_{t+1} \dots s_{n-1}, \\ w_{n-1} = s_{t-1} s_t \dots s_{n-2}, \\ \dots \dots \dots \\ w_{r+2} = s_{t-n+r+2} s_{t-n+r+3} \dots s_{r+1}, \\ w_r = s_{t-n+r+1} s_{t-n+r+2} \dots s_{r-1}, \end{cases}$$

and by direct calculation we see that such  $d(t)$  is the last Garnir tableau of shape  $\lambda$ .  $\square$

**Example 4.69.** Suppose  $\lambda = (7, 5, 3)$  and  $t = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 8 & 9 & 12 & 14 & 15 & & \\ \hline 10 & 11 & 13 & & & & \\ \hline \end{array}$ . Then

$$d(t) = s_{12} s_{13} s_{14} \cdot s_{11} s_{12} s_{13} \cdot s_{10} s_{11}.$$

So we can write  $d(t) = w_{15} w_{14} w_{13} w_{12}$  where  $w_{15} = s_{12} s_{13} s_{14}$ ,  $w_{14} = s_{11} s_{12} s_{13}$ ,  $w_{13} = 1$  and  $w_{12} = s_{10} s_{11}$ . Notice  $l(w_{15}) = l(w_{14}) = l(w_{12}) + 1$  and  $t \cdot s_{12}$  is not standard, and furthermore,  $t$  is the last Garnir tableau of shape  $\lambda$ .

Finally we are ready to prove the most important result of this subsection.

**Proposition 4.70.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$  and that  $t$  is a standard  $\lambda$ -tableau such that  $l(d(t)) \leq m_\lambda$  and  $d(t) \cdot s_r$  is not reduced or  $t \cdot s_r$  is not standard for some  $r$ . Then

$$\psi_{\text{st}} \psi_r = \sum_{(u,v) \triangleright (s,t)} c_{uv} \psi_{uv}.$$

for any standard  $\lambda$ -tableau  $s$ .

*Proof.* First we set  $s = t^\lambda$ . By the definition of  $m_\lambda$  the Proposition holds when  $l(d(t)) < m_\lambda$ . Hence we only have to consider the case when  $l(d(t)) = m_\lambda$ . By Corollary 4.4 we have  $l(d(t)) > 0$ . Therefore  $\psi_{d(t)} \neq 1$ .

Recall that the standard expression of  $\psi_{d(t)}$  has the form  $\psi_{w_n} \psi_{w_{n-1}} \dots \psi_{w_2}$  where  $\psi_{w_i} = \psi_{a_i} \psi_{a_i+1} \psi_{a_i+2} \dots \psi_{i-1}$  for some  $a_i \leq i-1$  or  $\psi_{w_i} = 1$ . Let  $k$  be the smallest positive integer such that  $\psi_{w_k} \neq 1$  and  $\psi_{w_i} = 1$  for all  $i < k$ . Because  $\psi_{d(t)} \neq 1$ , the integer  $k$  is well-defined. So  $\psi_{d(t)} = \psi_{w_n} \psi_{w_{n-1}} \dots \psi_{w_k}$ .

Recall that by Lemma 4.55 and 4.59, if  $d(t)$  is unlocked by  $s_r$  in type I or type II, we have  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$ .

We separate the problem into several cases:

**Case 4.70a:**  $k-1 \notin \{r-1, r, r+1\}$ . Then

$$\psi_{d(t)} = \psi_{w_n} \dots \psi_{w_k} = \psi_{w_n} \dots \psi_{w_{k+1}} \psi_{a_k} \psi_{a_k+1} \dots \psi_{k-2} \psi_{k-1}.$$

In this case  $t$  is unlocked by  $s_r$  in type I. Therefore by Lemma 4.55,  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$ .

**Case 4.70b:**  $k-1 = r$ . Define  $w = t \cdot s_r$ . Hence  $d(t) = d(w) \cdot s_r$ . Write  $\mathbf{i}_w = (i_1 i_2 \dots i_n)$ .

$$e_\lambda y_\lambda \psi_{d(t)} \psi_r = y_\lambda \psi_{d(w)} e(\mathbf{i}_w) \psi_r^2 = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ y_\lambda \psi_{d(w)} e(\mathbf{i}_w) = \psi_{t^\lambda} \psi_r, & \text{if } |i_r - i_{r+1}| > 1, \\ \pm y_\lambda \psi_{d(w)} e(\mathbf{i}_w) (y_r - y_{r+1}) \\ = \pm \psi_{t^\lambda} \psi_r (y_r - y_{r+1}), & \text{if } i_r = i_{r+1} \pm 1. \end{cases}$$

By Proposition 4.9 we have  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$ .

**Case 4.70c:**  $k-1 = r+1$ .

**4.70c.1:**  $\psi_{w_i} = 1$  for some  $i > k$  and  $i \neq n$ . Then we have

$$\begin{aligned} \psi_{d(t)} \psi_r &= \psi_{w_n} \psi_{w_{n-1}} \dots \psi_{w_{i+2}} \psi_{w_{i+1}} \psi_{w_{i-1}} \dots \psi_{w_k} \psi_r \\ &= \psi_{w_n} \psi_{w_{n-1}} \dots \psi_{w_{i+2}} (\psi_{a_{i+1}} \psi_{a_{i+1}+1} \dots \psi_{i-1} \psi_i) \psi_{w_{i-1}} \dots \psi_{w_k} \psi_r. \end{aligned}$$

As  $i > k = r+2 > r+1$ , we have

$$\psi_{d(t)} \psi_r = (\psi_{w_n} \psi_{w_{n-1}} \dots \psi_{w_{i+2}} \psi_{a_{i+1}} \psi_{a_{i+1}+1} \dots \psi_{i-1} \psi_{w_{i-1}} \dots \psi_{w_k} \psi_i) \psi_r,$$

which shows that  $t$  is unlocked by  $s_r$  in type I. By Lemma 4.55 we have  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$ .

**4.70c.2:**  $\psi_{w_n} = 1$ . In this case  $\psi_{n-1}$  is not involved in  $\psi_{d(t)} \psi_r$ . By Lemma 4.15 we have  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$ .

**4.70c.3:**  $\psi_{w_i} \neq 1$  for  $i > k$  and  $l(\psi_{w_k}) > 1$ . Then we see that  $t$  is unlocked by  $s_r$  in type II. By Lemma 4.59 we have  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$ .

**4.70c.4:**  $\psi_{w_i} \neq 1$  for  $i > k$ , and there exists  $k < j < n$  such that  $l(\psi_{w_k}) = l(\psi_{w_{k+1}}) = \dots = l(\psi_{w_{j-1}}) = 1$  and  $l(\psi_{w_j}) > 1$ . Then we have

$$w_j \cdot w_{j-1} = s_{a_j} s_{a_j+1} \dots s_{j-3} s_{j-2} s_{j-1} \cdot s_{j-2} = s_{a_j} s_{a_j+1} \dots s_{j-3} \cdot s_{j-1} s_{j-2} s_{j-1}.$$

Therefore

$$\begin{aligned} d(t) &= w_n w_{n-1} \dots w_{j+1} \cdot s_{a_j} s_{a_j+1} \dots s_{j-3} \cdot s_{j-1} s_{j-2} s_{j-1} \cdot w_{j-2} \dots w_k \\ &= w_n w_{n-1} \dots w_{j+1} s_{a_j} s_{a_j+1} \dots s_{j-3} \cdot s_{j-1} s_{j-2} w_{j-2} \dots w_k \cdot s_{j-1}, \end{aligned}$$

and  $j-1 \geq k = r+2 > r+1$ ,  $s_{j-1}$  and  $s_r$  commute, which shows that  $t$  is unlocked by  $s_r$  in type I. By Lemma 4.55, we have  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$ .

**4.70c.5:**  $l(\psi_{w_k}) = l(\psi_{w_{k+1}}) = \dots = l(\psi_{w_n}) = 1$ . Then by Lemma 4.63,  $t$  is the last Garnir tableau of shape  $\lambda$ . Hence by Proposition 4.25,  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$ .

**Case 4.70d:**  $k-1 = r-1$ .

**4.70d.1:**  $\psi_{w_{k+1}} \neq 1$ . Then

$$\begin{aligned} d(t) &= w_n w_{n-1} \dots w_{k+2} \cdot w_{k+1} w_k \\ &= w_n w_{n-1} \dots w_{k+2} \cdot s_{a_{k+1}} s_{a_{k+1}+1} \dots s_{k-1} s_k \cdot s_{a_k} s_{a_k+1} \dots s_{k-2} s_{k-1} \\ &= w_n w_{n-1} \dots w_{k+2} s_{a_{k+1}} s_{a_{k+1}+1} \dots s_{k-1} s_{a_k} s_{a_k+1} \dots s_{k-2} \cdot s_k s_{k-1}, \end{aligned}$$

and as  $r = k$ , we see that  $t$  is unlocked by  $s_r$  in type II. Therefore  $\psi_{t^\lambda} \psi_r \in R_n^{\geq \lambda}$  by Lemma 4.59.

**4.70d.2:**  $k = n - 1$  and  $\psi_{w_{k+1}} = \psi_{w_n} = 1$ . The  $\psi_{d(t)}\psi_r = \psi_{w_{n-1}}\psi_{n-1} = \psi_{a_{n-1}}\psi_{a_{n-1}+1} \dots \psi_{n-2}\psi_{n-1}$ . Then by Lemma 4.65,  $t$  is the last Garnir tableau of shape  $\lambda$ . Hence by Proposition 4.25,  $\psi_{t^l t}\psi_r \in R_n^{\geq \lambda}$ .

**4.70d.3:**  $k < n - 1$ ,  $\psi_{w_{k+1}} = 1$  and  $\psi_{w_n} = 1$ . Then  $n - 1 > k = r$ . So  $\psi_{d(t)}\psi_r$  doesn't involve  $\psi_{n-1}$ . By Lemma 4.15 we have  $\psi_{t^l t}\psi_r \in R_n^{\geq \lambda}$ .

**4.70d.4:**  $k < n - 1$ ,  $\psi_{w_{k+1}} = 1$  and there exists  $k + 1 < j < n$  such that  $\psi_{w_j} = 1$  and  $\psi_{w_{j+1}} \neq 1$ . In this case we have

$$\begin{aligned} d(t) &= w_n w_{n-1} \dots w_{j+2} w_{j+1} w_j w_{j-1} \dots w_k \\ &= w_n w_{n-1} \dots w_{j+2} \cdot s_{a_{j+1}} s_{a_{j+1}+1} \dots s_{j-1} s_j \cdot w_{j-1} \dots w_k \\ &= (w_n w_{n-1} \dots w_{j+2} \cdot s_{a_{j+1}} s_{a_{j+1}+1} \dots s_{j-1} \cdot w_{j-1} \dots w_k) \cdot s_j. \end{aligned}$$

As  $j > k + 1 = r + 1$ ,  $\psi_j$  and  $\psi_r$  commute. Therefore  $t$  is unlocked by  $s_r$  in type I. By Lemma 4.55, we have  $\psi_{t^l t}\psi_r \in R_n^{\geq \lambda}$ .

**4.70d.5:**  $k < n - 1$ ,  $\psi_{w_{k+1}} = 1$  and for any  $j > k + 1$ ,  $\psi_{w_j} \neq 1$ . Then by Lemma 4.67, we have  $l(\psi_{w_j}) \geq l(\psi_{w_k}) + 1$  for all  $j \geq k + 2$ .

**4.70d.5.1:** Suppose  $l(\psi_{w_{k+2}}) > l(\psi_{w_k}) + 1$ . So we have  $a_{k+2} \leq a_k$ , and therefore

$$\begin{aligned} w_{k+2} w_k &= s_{a_{k+2}} s_{a_{k+2}+1} \dots s_k s_{k+1} \cdot s_{a_k} s_{a_k+1} \dots s_{k-2} s_{k-1} \\ &= s_{a_k+1} \dots s_{k-1} s_k \cdot s_{a_{k+2}} s_{a_{k+2}+1} \dots s_k s_{k+1}. \end{aligned}$$

Therefore

$$\begin{aligned} d(t) &= w_n w_{n-1} \dots w_{k+3} \cdot w_{k+2} w_k \\ &= w_n w_{n-1} w_{k+3} \cdot s_{a_k+1} \dots s_{k-1} s_k \cdot s_{a_{k+2}} s_{a_{k+2}+1} \dots s_k s_{k+1}. \end{aligned}$$

Then because  $k = r$ ,  $t$  is unlocked by  $s_r$  in type II. Therefore, by Lemma 4.59,  $\psi_{t^l t}\psi_r \in R_n^{\geq \lambda}$ .

**4.70d.5.2:** There exists  $j > k + 2$  such that  $l(\psi_{w_{k+2}}) = l(\psi_{w_{k+3}}) = \dots = l(\psi_{w_{j-1}}) = l(\psi_{w_k}) + 1$  and  $l(\psi_{w_j}) > l(\psi_{w_k}) + 1$ . So we have  $l(\psi_{w_j}) > l(\psi_{w_{j-1}})$  and  $a_j \leq a_{j-1}$ , and therefore

$$\begin{aligned} w_j w_{j-1} &= s_{a_j} s_{a_j+1} \dots s_{j-2} s_{j-1} \cdot s_{a_{j-1}} s_{a_{j-1}+1} \dots s_{j-3} s_{j-2} \\ &= s_{a_{j-1}+1} \dots s_{j-2} s_{j-1} \cdot s_{a_j} s_{a_j+1} \dots s_{j-2} s_{j-1}. \end{aligned}$$

Therefore

$$\begin{aligned} d(t) &= w_n w_{n-1} \dots w_{j+1} w_j w_{j-1} w_{j-2} \dots w_k \\ &= w_n w_{n-1} \dots w_{j+1} \cdot s_{a_{j-1}+1} \dots s_{j-1} \cdot s_{a_j} \dots s_{j-2} s_{j-1} \cdot w_{j-2} \dots w_k \\ &= (w_n w_{n-1} \dots w_{j+1} \cdot s_{a_{j-1}+1} \dots s_{j-1} \cdot s_{a_j} \dots s_{j-2} \cdot w_{j-2} \dots w_k) \cdot s_{j-1}. \end{aligned}$$

Then because  $j - 1 > k + 1 = r + 1$ ,  $s_{j-1}$  and  $s_r$  commutes. Hence  $t$  is unlocked by  $s_r$  in type I and therefore, by Lemma 4.55,  $\psi_{t^l t}\psi_r \in R_n^{\geq \lambda}$ .

**4.70d.5.3:**  $l(\psi_{w_{k+2}}) = l(\psi_{w_{k+3}}) = \dots = l(\psi_{w_{n-1}}) = l(\psi_{w_k}) + 1$ . By Lemma 4.68,  $t$  is the last Garnir tableau of shape  $\lambda$ . By Proposition 4.25, we have  $\psi_{t^l t}\psi_r \in R_n^{\geq \lambda}$ .

By the above cases,  $\psi_{t^l t}\psi_r$  is always in  $R_n^{\geq \lambda}$ . Therefore by Proposition 2.41, we have

$$\psi_{t^l t}\psi_r = \sum_{(u,v) \triangleright (t^l, t)} c_{uv} \psi_{uv} = \sum_{v \triangleright t} c_{t^l v} \psi_{t^l v} + \sum_{u, v \in \text{Std}(> \lambda)} c_{uv} \psi_{uv}.$$

Giving any standard  $\lambda$ -tableau  $s$ , we have  $\psi_{st}\psi_r = \psi_{d(s)}^* \psi_{t^l t}\psi_r$ . Notice  $\psi_{d(s)}^* \psi_{t^l v} = \psi_{sv}$ . For any  $u, v \in \text{Std}(> \lambda)$ ,  $\psi_{uv} \in R_n^{\geq \lambda}$ . As  $\lambda \in \mathcal{S}_n^\Lambda$ , by Lemma 3.12,  $R_n^{\geq \lambda}$  is an ideal. Therefore  $\psi_{d(s)}^* \psi_{uv} \in R_n^{\geq \lambda}$ . These arguments yield that  $\psi_{st}\psi_r \in R_n^{\geq \lambda}$ . By Proposition 2.41 we completes the proof.  $\square$

The following Corollary is straightforward by Corollary 4.7 and Proposition 4.70.

**Corollary 4.71.** Suppose  $\lambda \in \mathcal{S}_n^\Lambda$ , for any standard  $\lambda$ -tableau  $t$  with  $l(d(t)) \leq m_\lambda$ , then

$$\psi_{st}\psi_r = \begin{cases} \psi_{t^l w} + \sum_{(u,v) \triangleright (s,t)} c_{uv} \psi_{uv}, & \text{if } w = u \cdot s_r \text{ is standard and } d(u) \cdot s_r \text{ is reduced,} \\ \sum_{(u,v) \triangleright (s,t)} c_{uv} \psi_{uv}, & \text{if } u \cdot s_r \text{ is not standard or } d(u) \cdot s_r \text{ is not reduced.} \end{cases}$$

for any standard  $\lambda$ -tableau  $\mathbf{s}$ .

#### 4.6. Integral basis Theorem

In this subsection we will complete the main Theorem of this paper.

**Theorem 4.72.** *Suppose  $\lambda \in \mathcal{S}_n^\Lambda$ , we have  $\lambda \in \mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$ .*

*Proof.* By Theorem 3.8 we have when  $\lambda \in \mathcal{S}_n^\Lambda$  then  $\lambda \in \mathcal{P}_I^\Lambda$ . By Corollary 4.4, we have  $0 < m_\lambda$ , i.e.  $1 \leq m_\lambda$ . Assume  $l = l(d(\mathbf{u}))$  for some  $\mathbf{u} \in \text{Std}(\lambda)$ , by Proposition 4.9 and Corollary 4.71, for any  $\mathbf{t} \in \text{Std}(\lambda)$  with  $l(d(\mathbf{t})) = l$ , we have

$$\begin{aligned} \psi_{\mathbf{st}} \psi_r &= \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})} c_{\mathbf{uv}} \psi_{\mathbf{uv}}, \\ \psi_{\mathbf{st}} \psi_r &= \begin{cases} \psi_{\mathbf{sw}} + \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})} c_{\mathbf{uv}} \psi_{\mathbf{uv}}, & \text{if } \mathbf{w} = \mathbf{t} \cdot s_r \text{ is standard and } d(\mathbf{u}) \cdot s_r \text{ is reduced,} \\ \sum_{(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})} c_{\mathbf{uv}} \psi_{\mathbf{uv}}, & \text{if } \mathbf{u} \cdot s_r \text{ is not standard or } d(\mathbf{u}) \cdot s_r \text{ is not reduced.} \end{cases} \end{aligned}$$

This implies that  $l < m_\lambda$ , i.e.  $l + 1 \leq m_\lambda$ . So by induction, for any  $\mathbf{t} \in \text{Std}(\lambda)$ , we have  $l(d(\mathbf{t})) < m_\lambda$ . Therefore  $\lambda \in \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$ . This completes the proof.  $\square$

**Theorem 4.73.** *The set  $\{\psi_{\mathbf{st}}^\mathbb{Z} \mid \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n^\Lambda\}$  is a graded cellular basis of  $\mathcal{R}_n^\Lambda(\mathbb{Z})$ .*

*Proof.* It's trivial that when  $n = 1$  the Theorem holds. Assume for any  $n' < n$  the Theorem follows. Suppose we can write all multipartitions of  $n$  as  $\lambda_{[1]}, \lambda_{[2]}, \dots, \lambda_{[k]}$  where  $\lambda_{[1]} > \lambda_{[2]} > \dots > \lambda_{[k]}$ . As  $\lambda_{[1]} = ((n), \emptyset, \dots, \emptyset)$ , by Lemma 3.9, Corollary 3.11 and 3.10, we have  $\lambda_{[1]} \in \mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$ . Hence  $\lambda_{[2]} \in \mathcal{S}_n^\Lambda$ . Now assume  $\lambda_{[i]} \in \mathcal{S}_n^\Lambda$ , by Theorem 4.72,  $\lambda_{[i]} \in \mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$ . Hence  $\lambda_{[i+1]} \in \mathcal{S}_n^\Lambda$ . Therefore for any  $i$ ,  $\lambda_{[i]} \in \mathcal{S}_n^\Lambda$ . Hence, for any  $\lambda \in \mathcal{P}_n^\Lambda$ ,  $\lambda \in \mathcal{P}_I^\Lambda \cap \mathcal{P}_y^\Lambda \cap \mathcal{P}_\psi^\Lambda$ . Recall that

$$R_n^\Lambda = \{r \in \mathcal{R}_n^\Lambda(\mathbb{Z}) \mid r = \sum_{\substack{\mathbf{s}, \mathbf{t} \in \text{Std}(\mu) \\ \mu \in \mathcal{P}_n^\Lambda}} c_{\mathbf{st}} \psi_{\mathbf{st}}, c_{\mathbf{st}} \in \mathbb{Z}\}.$$

So  $R_n^\Lambda$  is an ideal.

Now, for any  $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$ , set  $\mathbf{j} = (i_1, i_2, \dots, i_{n-1}) \in I^{n-1}$ . Because  $e(\mathbf{j}) \in \mathcal{R}_{n-1}^\Lambda$ , by assumption we have  $e(\mathbf{j}) = \sum_{\substack{\mu \in \mathcal{P}_{n-1}^\Lambda \\ \mathbf{u}, \mathbf{v} \in \text{Std}(\mu)}} c_{\mathbf{uv}} \psi_{\mathbf{uv}} \in R_{n-1}^\Lambda$  and hence that  $e(\mathbf{i}) = \theta_{i_n}(e(\mathbf{j})) = \sum_{\substack{\mu \in \mathcal{P}_{n-1}^\Lambda \\ \mathbf{u}, \mathbf{v} \in \text{Std}(\mu)}} c_{\mathbf{uv}} \theta_{i_n}(\psi_{\mathbf{uv}})$ .

For any  $\mu \in \mathcal{P}_{n-1}^\Lambda$  and  $\mathbf{u}, \mathbf{v} \in \text{Std}(\mu)$ , we have

$$\theta_{i_n}(\psi_{\mathbf{uv}}) = \psi_{d(\mathbf{u})}^* e(\mathbf{i}_\mu \vee i_n) y_\mu \psi_{d(\mathbf{v})}.$$

By Lemma 3.3 and Theorem 3.8, we have  $e(\mathbf{i}_\mu \vee i_n) y_\mu \psi_{d(\mathbf{v})}^0 \in R_n^\Lambda$ . Then because  $R_n^\Lambda$  is an ideal,

$$e(\mathbf{i}) = \sum_{\substack{\mu \in \mathcal{P}_{n-1}^\Lambda \\ \mathbf{u}, \mathbf{v} \in \text{Std}(\mu)}} c_{\mathbf{uv}} \theta_{i_n}(\psi_{\mathbf{uv}}) \in R_n^\Lambda.$$

Then we have  $R_n^\Lambda = \mathcal{R}_n^\Lambda(\mathbb{Z})$ . By Corollary 2.42, the set  $\{\psi_{\mathbf{st}}^\mathbb{Z} \mid \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n^\Lambda\}$  is linearly independent. Hence,  $\{\psi_{\mathbf{st}}^\mathbb{Z} \mid \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_n^\Lambda\}$  is a basis of  $\mathcal{R}_n^\Lambda(\mathbb{Z})$ .

By definition, the elements of the set are homogeneous. That this basis is cellular is trivial by Theorem 2.37 and Proposition 2.41. This completes the proof.  $\square$

The next Corollary is a straightforward application of Theorem 4.73.

**Corollary 4.74.** *For any  $\mathbf{i} \in I^n$ ,  $e(\mathbf{i}) \neq 0$  if and only if  $\mathbf{i}$  is the residue sequence of a standard tableau  $\mathbf{t}$ .*

*Proof.* Suppose  $\mathbf{i}$  is the residue sequence of a standard tableau  $\mathbf{t}$ . By Theorem 4.73 we have  $\psi_{\mathbf{tt}}^\mathbb{Z} \neq 0$ . Because  $\psi_{\mathbf{tt}}^\mathbb{Z} = \psi_{\mathbf{tt}}^\mathbb{Z} e(\mathbf{i})$ , we must have  $e(\mathbf{i}) \neq 0$ .

Suppose  $\mathbf{i}$  is not the residue sequence of any standard tableau. By Theorem 4.73 we can write

$$1 = \sum_{\mathbf{s}, \mathbf{t}} c_{\mathbf{st}} \psi_{\mathbf{st}}^\mathbb{Z},$$

and hence  $e(\mathbf{i}) = 1 \cdot e(\mathbf{i}) = \sum_{\mathbf{s}, \mathbf{t}} c_{\mathbf{st}} \psi_{\mathbf{st}}^\mathbb{Z} e(\mathbf{i}) = 0$ , which completes the proof.  $\square$

## 5. A new basis of the Affine KLR Algebras

In Theorem 4.73 we have shown that  $\mathcal{R}_n^\Lambda(\mathbb{Z})$  is a  $\mathbb{Z}$ -free algebra with basis  $\{\psi_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n^\Lambda\}$ . In this section we extend this result and construct a graded cellular basis of  $\mathcal{R}_n^\Lambda(\mathbb{Z})$ . Moreover, for any weight  $\Lambda$  we obtain a homogeneous basis of the ideal  $N_n^\Lambda$  as a subset of the graded cellular basis of  $\mathcal{R}_n^\Lambda(\mathbb{Z})$ . As an application we obtain a new classification of the complete set of simple  $\mathcal{R}_n^\Lambda$ -modules by using the graded cellular basis of  $\mathcal{R}_n^\Lambda$ . Unlike in previous sections, this section laces no constraints upon  $e$  as we now allow the case  $e = 2$ .

### 5.1. Infinite sequence of weights and basis of $\mathcal{R}_\alpha^\Lambda$

This subsection introduces a special kind of sequences of weights  $\Lambda^\infty$ . We use these infinite sequences of weights to define a graded cellular basis for  $\mathcal{R}_\alpha^\Lambda$  by combining the graded cellular basis of the algebras  $\mathcal{R}_\alpha$ .

We fix an integer  $e \geq 0$  and  $e \neq 1$ , and consider the KLR algebras  $\mathcal{R}_n = \mathcal{R}_n(\mathbb{Z})$ . Suppose  $\Lambda = \sum_{i \in I} a_i \Lambda_i$  and  $\Lambda' = \sum_{i \in I} a'_i \Lambda_i$  are two weights in  $P_+$ . Define a partial ordering on  $P_+$  and write  $\Lambda \leq \Lambda'$  if  $a_i \leq a'_i$  for all  $i \in I$ . Write  $\Lambda < \Lambda'$  if  $\Lambda \leq \Lambda'$  and  $\Lambda \neq \Lambda'$ .

**Definition 5.1.** Suppose  $\Lambda^\infty = (\Lambda^{(k)})_{k \geq 1}$  is a sequence of weights in  $P_+$  where  $\Lambda^{(k)} = \sum_{i \in I} a_i^{(k)} \Lambda_i$ . It is an **increasing sequence** if  $\Lambda^{(k)} < \Lambda^{(k+1)}$  for all  $k \geq 1$ . The sequence  $\Lambda^\infty$  is called **standard** if  $\lim_{k \rightarrow \infty} a_i^{(k)} = \infty$ , for all  $i \in I$ .

**Example 5.2.** Suppose  $e > 0$ . Let  $\Lambda^\infty$  be any increasing sequence of weights such that  $\Lambda^{(1)} = \Lambda_1$  and  $\Lambda^{(k)} = \Lambda^{(k-1)} + \Lambda_i$  whenever  $k \equiv i \pmod{e}$ . For example, when  $e = 3$ , we have

$$\begin{aligned} \Lambda^{(1)} &= \Lambda_1, \\ \Lambda^{(2)} &= \Lambda_1 + \Lambda_2, \\ \Lambda^{(3)} &= \Lambda_1 + \Lambda_2 + \Lambda_0, \\ \Lambda^{(4)} &= 2\Lambda_1 + \Lambda_2 + \Lambda_0, \\ \Lambda^{(5)} &= 2\Lambda_1 + 2\Lambda_2 + \Lambda_0, \\ \Lambda^{(6)} &= 2\Lambda_1 + 2\Lambda_2 + 2\Lambda_0, \\ \Lambda^{(7)} &= 3\Lambda_1 + 2\Lambda_2 + 2\Lambda_0, \\ &\dots \dots \dots \end{aligned}$$

So in this case we have  $\lim_{k \rightarrow \infty} a_i^{(k)} = \infty$  for any  $i \in I$  and  $\Lambda^\infty$  is a standard sequence.

**Example 5.3.** Suppose  $e = 0$ . Define  $\Lambda^\infty$  where  $\Lambda^{(1)} = \Lambda_0$  and  $\Lambda^{(k)} = \Lambda^{(k-1)} + \Lambda_i$  with  $i = (k-1) - (n-1)^2 - (n-1) = k - n^2 + n - 1$  if  $(n-1)^2 < k \leq n^2$ . In more details,

$$\begin{aligned} \Lambda^{(1)} &= \Lambda_0, \\ \Lambda^{(2)} &= \Lambda_{-1} + \Lambda_0, \\ \Lambda^{(3)} &= \Lambda_{-1} + 2\Lambda_0, \\ \Lambda^{(4)} &= \Lambda_{-1} + 2\Lambda_0 + \Lambda_1, \\ \Lambda^{(5)} &= \Lambda_{-2} + \Lambda_{-1} + 2\Lambda_0 + \Lambda_1, \\ \Lambda^{(6)} &= \Lambda_{-2} + 2\Lambda_{-1} + 2\Lambda_0 + \Lambda_1, \\ \Lambda^{(7)} &= \Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + \Lambda_1, \\ \Lambda^{(8)} &= \Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + 2\Lambda_1, \\ \Lambda^{(9)} &= \Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + 2\Lambda_1 + \Lambda_2, \\ \Lambda^{(10)} &= \Lambda_{-3} + \Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + 2\Lambda_1 + \Lambda_2, \\ \Lambda^{(11)} &= \Lambda_{-3} + 2\Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + 2\Lambda_1 + \Lambda_2, \\ &\dots \dots \dots \end{aligned}$$

So in this case we have  $\lim_{k \rightarrow \infty} a_i^{(k)} = \infty$  for any  $i \in I$  and  $\Lambda^\infty$  is a standard sequence.

Recall that for any weight  $\Lambda = \sum_{i \in I} a_i \Lambda_i$ , we can define the two-sided ideal  $N_n^\Lambda$  of  $\mathcal{R}_n^\Lambda$ . The ideal  $N_n^\Lambda$  is generated by elements  $e(\mathbf{i})y_1^{a_{i_1}}$  for all  $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$ . By definition,  $\mathcal{R}_n^\Lambda \cong \mathcal{R}_n / N_n^\Lambda$ . There is a natural injective homomorphism sending  $\mathcal{R}_n^\Lambda$  to  $\mathcal{R}_n$  by sending  $e(\mathbf{i})$ ,  $y_r$  and  $\psi_r$  to  $\hat{e}(\mathbf{i})$ ,  $\hat{y}_r$  and  $\hat{\psi}_r$ , respectively. Hence we can consider  $\mathcal{R}_n^\Lambda$  as a  $\mathbb{Z}$ -submodule of  $\mathcal{R}_n$  and write  $\mathcal{R}_n \cong \mathcal{R}_n^\Lambda \oplus N_n^\Lambda$  as  $\mathbb{Z}$ -modules.

Recall  $Q_+ = \sum_{i \in I} \mathbb{N} \alpha_i$  is defined in subsection 1.1. For  $\alpha = \sum_{i \in I} a_i \alpha_i \in Q_+$ , define  $|\alpha| = \sum_{i \in I} a_i$ . Then for any  $\alpha \in Q_+$  with  $|\alpha| = n$ , define  $I^\alpha$  to be the set of all sequences  $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$  such that  $a_i =$

$|\{1 \leq r \leq n \mid i_r = i\}|$ . By definition, if  $\mathbf{i}, \mathbf{j} \in I^\alpha$  then there exists  $v \in \mathfrak{S}_n$  such that  $\mathbf{i} = \mathbf{j} \cdot v$ . Define  $\hat{e}_\alpha = \sum_{\mathbf{i} \in I^\alpha} \hat{e}(\mathbf{i}) \in \mathcal{R}_n$  and  $e_\alpha = \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i}) \in \mathcal{R}_n^\Lambda$ .

The following result is trivial by the relations of  $\mathcal{R}_n$ .

**Lemma 5.4.** *Suppose  $\alpha, \beta \in Q_+$ . Then  $\mathcal{R}_n \hat{e}_\alpha \neq 0$  and  $\hat{e}_\beta \mathcal{R}_n \hat{e}_\alpha = \delta_{\alpha\beta} \mathcal{R}_n e_\alpha = \delta_{\alpha\beta} \hat{e}_\beta \mathcal{R}_n$ .*

We then define  $\mathcal{R}_\alpha = \mathcal{R}_n \hat{e}_\alpha$ ,  $\mathcal{R}_\alpha^\Lambda = \mathcal{R}_n^\Lambda e_\alpha$  and  $N_\alpha^\Lambda = N_n^\Lambda \hat{e}_\alpha$ . We see that  $\mathcal{R}_\alpha \hat{e}(\mathbf{j}) = 0$  if  $\mathbf{j} \notin I^\alpha$ . Finally, because

$$\mathcal{R}_n = \bigoplus_{\alpha \in Q_+} \mathcal{R}_\alpha \quad \text{and} \quad \mathcal{R}_n^\Lambda = \bigoplus_{\alpha \in Q_+} \mathcal{R}_\alpha^\Lambda,$$

and by the relations  $\mathcal{R}_\alpha$  and  $\mathcal{R}_\alpha^\Lambda$ 's are subalgebras of  $\mathcal{R}_n$  and  $\mathcal{R}_n^\Lambda$ , respectively. Hence we will mainly work in  $\mathcal{R}_\alpha$ ,  $\mathcal{R}_\alpha^\Lambda$  and  $N_\alpha^\Lambda$  and extend the basis of  $\mathcal{R}_\alpha^\Lambda$  to  $\mathcal{R}_\alpha$  and hence generate a graded cellular basis of  $\mathcal{R}_n$ .

By Theorem 4.73 and the orthogonality of  $e(\mathbf{i})$ 's we can give a basis for  $\mathcal{R}_\alpha^\Lambda$ .

**Proposition 5.5.** *Suppose  $\mathbf{i} \in I^n$  and  $\Lambda \in P_+$ . The set*

$$\{\psi_{\text{st}} \mid \lambda \in \mathcal{P}_n^\Lambda, \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda), \text{res}(\mathbf{t}) \in I^\alpha\}$$

*is a graded cellular basis of  $\mathcal{R}_\alpha^\Lambda$ .*

## 5.2. Minimum degree of $N_\alpha^\Lambda$

Fix  $\alpha \in Q_+$ . In the last subsection we introduced a standard sequence of  $\Lambda^\infty$ . For each  $k$  and  $\mathbf{i} \in I^n$ , we define the **minimum degree** of  $N_\alpha^{\Lambda^{(k)}}$  to be the integer

$$m_\alpha^{\Lambda^{(k)}} = \min\{\deg(r) \mid r \text{ is a homogeneous element in } N_\alpha^{\Lambda^{(k)}}\}.$$

We will prove that  $m_\alpha^{\Lambda^{(k)}} \rightarrow \infty$  with  $k \rightarrow \infty$ . This result is quite important in the next subsection because it will allow us to extend the basis of  $\mathcal{R}_\alpha^\Lambda$  to  $\mathcal{R}_\alpha$ .

First we need to find a general description of the homogeneous elements of  $N_\alpha^\Lambda$ .

**Lemma 5.6.** *For  $\Lambda = \sum_{i \in I} a_i \Lambda_i \in P_+$  and  $\alpha \in Q_+$ , the ideal  $N_\alpha^\Lambda$  is spanned by*

$$\{\psi_u e(\mathbf{i}) y_1^{a_{i_1}} f(y) \psi_v \mid u, v \in \mathfrak{S}_n, f(y) \in \mathbb{Z}[y_1, y_2, \dots, y_n], \mathbf{i} \in I^\alpha\}.$$

*Proof.* By the definition of  $N_\alpha^\Lambda$ , any element of  $N_\alpha^\Lambda$  can be written as linear combination of elements of the form

$$\psi_{u_k} f_k(y) \psi_{u_{k-1}} \dots \psi_{u_2} f_2(y) \psi_{u_1} f_1(y) e(\mathbf{i}) y_1^{a_{i_1}} g_1(y) \psi_{v_1} g_2(y) \psi_{v_2} \dots \psi_{v_{l-1}} g_l(y) \psi_{v_l}, \quad (5.7)$$

where  $u_i, v_i \in \mathfrak{S}_n$ ,  $\mathbf{i} \in I^\alpha$  and  $f_i(y), g_i(y) \in \mathbb{Z}[y_1, \dots, y_n]$ . In the view of Lemma 4.6 and [5, Proposition 2.5], every element in the form of (5.7) can be written as linear combination of terms of the form  $\psi_u e(\mathbf{i}) y_1^{a_{i_1}} f(y) \psi_v$ 's. Hence  $N_\alpha^\Lambda$  is spanned by the elements given in the statement of the Lemma.  $\square$

The next result is directly implied by the above Lemma.

**Proposition 5.8.** *Suppose that  $\Lambda^\infty$  is a standard sequence and  $\alpha \in Q_+$ . Then  $\lim_{k \rightarrow \infty} m_\alpha^{\Lambda^{(k)}} = \infty$ .*

*Proof.* By Lemma 5.6, for any  $k \geq 1$  we have

$$m_\alpha^{\Lambda^{(k)}} = \min\{\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}^{(k)}} f(y) \psi_v) \mid u, v \in \mathfrak{S}_n, f(y) \in \mathbb{Z}[y_1, y_2, \dots, y_n], \mathbf{i} \in I^\alpha\}.$$

By definition,  $\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}^{(k)}} f(y) \psi_v) = \deg(\psi_u e(\mathbf{i})) + \deg(y_1^{a_{i_1}^{(k)}}) + \deg(f(y)) + \deg(\psi_v e(\mathbf{i} \cdot v))$ . As  $u$  and  $v$  are reduced expressions of permutations in  $\mathfrak{S}_n$ ,  $l(u) \leq \frac{(n-1)n}{2}$ , and  $\deg(\psi_v e(\mathbf{i})) \geq -2$  for any  $\mathbf{i}$ . Hence,  $\deg(\psi_u e(\mathbf{i})) \geq -(n-1)n$ .

By the same reasoning,  $\deg(\psi_v e(\mathbf{i} \cdot v)) \geq -(n-1)n$ . Then as  $\deg(f(y)) \geq 0$ , we have  $\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}^{(k)}} f(y) \psi_v) \geq -2(n-1)n + 2a_{i_1}^{(k)}$ .

Define  $a_\alpha^{(k)} = \min_{\mathbf{i} \in I^\alpha} a_{i_1}^{(k)}$ . We have

$$\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}^{(k)}} f(y) \psi_v) \geq -2(n-1)n + 2a_\alpha^{(k)},$$

for any  $u, v$  and  $f$ . Therefore  $m_\alpha^{\Lambda^{(k)}} \geq 2a_\alpha^{(k)} - 2(n-1)n$ .

Choose  $\mathbf{j} \in I^\alpha$ . By definition,  $I^\alpha = \{\mathbf{i} \in I^n \mid \mathbf{i} = \mathbf{j} \cdot v \text{ with } v \in \mathfrak{S}_n\}$ . Then  $|I^\alpha| \leq |\mathfrak{S}_n| < \infty$ . Then  $a_{i_1}^{(k)} \rightarrow \infty$  as  $k \rightarrow \infty$  for any  $\mathbf{i} \in I^\alpha$  implies  $a_\alpha^{(k)} \rightarrow \infty$  as  $k \rightarrow \infty$  because  $I^\alpha$  is finite. Therefore  $m_\alpha^{\Lambda^{(k)}} \rightarrow \infty$ .  $\square$

**Remark 5.9.** That the set  $I^\alpha$  is finite is important in the proof of Proposition 5.8. If it were possible for  $I^\alpha$  to be infinite then knowing that  $a_{i_1}^{(k)} \rightarrow \infty$  for all  $i \in I^\alpha$  is not strong enough to imply that  $a_\alpha^{(k)} \rightarrow \infty$  as  $k \rightarrow \infty$ .



### 5.3. A graded cellular basis of $\mathcal{R}_n$

In this subsection we will prove the main result of this section. First we introduce a special kind of multicharge  $\kappa$  corresponding to a standard sequence  $\Lambda^\infty$ . Then for any  $\alpha \in Q_+$ , we will find a graded cellular basis  $\mathcal{B}_\alpha^{\Lambda^\infty}$  of  $\mathcal{R}_\alpha$  that corresponds to  $\kappa$ .

**Definition 5.10.** Suppose  $\Lambda^\infty$  is a standard sequence. An **inverse multicharge sequence** for  $\Lambda^\infty$  is a infinite sequence  $\kappa = (\dots, \kappa_3, \kappa_2, \kappa_1)$  such that for any  $k \geq 1$ , if  $\ell_k = l(\Lambda^{(k)})$ , then  $\kappa_{\Lambda^{(k)}} = (\kappa_{\ell_k}, \kappa_{\ell_k-1}, \dots, \kappa_2, \kappa_1)$  is a multicharge corresponding to  $\Lambda^{(k)}$ .

**Example 5.11.** Suppose  $e = 3$ . Using the standard sequence  $\Lambda^\infty$  introduced in Example 5.2, we define a multicharge  $\kappa = (\dots, \kappa_3, \kappa_2, \kappa_1)$  where  $\kappa_k \equiv k \pmod{e}$  for  $k \geq 1$ . We can write  $\kappa = (\dots, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1)$ , and we have

$$\begin{aligned} \kappa_{\Lambda^{(1)}} &= (1), \\ \kappa_{\Lambda^{(2)}} &= (2, 1), \\ \kappa_{\Lambda^{(3)}} &= (0, 2, 1), \\ \kappa_{\Lambda^{(4)}} &= (1, 0, 2, 1), \\ \kappa_{\Lambda^{(5)}} &= (2, 1, 0, 2, 1), \\ &\dots \dots \dots \end{aligned}$$

All of these multicharges correspond to  $\Lambda^{(k)}$ .

Fix a standard sequence  $\Lambda^\infty$  and an inverse multicharge sequence  $\kappa$  for  $\Lambda^\infty$ . An **affine multipartition** of  $n$  is an ordered sequence  $\hat{\lambda} = (\dots, \lambda^{(2)}, \lambda^{(1)})$  of partitions such that  $\sum_{i=1}^\infty |\lambda^{(i)}| = n$ . Let  $\mathcal{P}_n^\kappa$  be the set of all affine multipartitions of  $n$ . We define **young diagram**  $[\hat{\lambda}]$  and **standard affine tableau**  $\hat{s}$  for affine multipartitions in the same way as for multipartitions. Let  $\text{Std}(\hat{\lambda})$  be the set of all standard affine tableaux of shape  $\hat{\lambda}$ .

Fix an inverse multicharge sequence  $\kappa = (\dots, \kappa_2, \kappa_1)$ . For any  $\ell > 0$ , let  $(\kappa_\ell, \dots, \kappa_1)$  be a multicharge and  $\Lambda$  be the unique weight corresponds to  $(\kappa_\ell, \dots, \kappa_1)$ . We define a map  $p_\ell: \mathcal{P}_n^\kappa \rightarrow \mathcal{P}_n^\Lambda$ , sending  $\hat{\lambda} = (\dots, \lambda^{(2)}, \lambda^{(1)}) \in \mathcal{P}_n^\kappa$  to  $\lambda = (\lambda^{(\ell)}, \lambda^{(\ell-1)}, \dots, \lambda^{(2)}, \lambda^{(1)}) \in \mathcal{P}_n^\Lambda$ .

Define the **level** of  $\hat{\lambda} \in \mathcal{P}_n^\kappa$  to be  $l(\hat{\lambda}) = \ell$  if  $\lambda^{(\ell)} \neq \emptyset$  and  $\lambda^{(i)} = \emptyset$  for  $i > \ell$ . Suppose  $\hat{\lambda} \in \mathcal{P}_n^\kappa$  with level  $\ell$  and  $\lambda = p_\ell(\hat{\lambda})$ , define  $q_\ell: \text{Std}(\hat{\lambda}) \rightarrow \text{Std}(\lambda)$  sending  $\hat{t} = (\dots, \hat{t}^{(2)}, \hat{t}^{(1)}) \in \text{Std}(\hat{\lambda})$  to  $t = (t^{(\ell)}, t^{(\ell-1)}, \dots, t^{(1)}) \in \text{Std}(\lambda)$ .

In order to simplify the notations, we write  $\lambda = p_\ell(\hat{\lambda})$  if  $l(\hat{\lambda}) = \ell$ . Similarly, if  $l(\hat{\lambda}) = \ell$ , for  $\hat{t} \in \text{Std}(\hat{\lambda})$ , we write  $t = q_\ell(\hat{t}) \in \text{Std}(\lambda)$ . Define the degree of each standard affine tableau to be  $\deg(\hat{s}) = \deg(s)$  and the residue sequence of the affine tableau  $\text{res}(\hat{s}) = \text{res}(s)$ .

Extend dominance ordering  $\succeq$  and lexicographic ordering  $\geq$  to  $\mathcal{P}_n^\kappa$  by defining  $\hat{\lambda} \succeq \hat{\mu}$  if  $l(\hat{\lambda}) > l(\hat{\mu})$  or  $l(\hat{\lambda}) = l(\hat{\mu})$  and  $\lambda \succeq \mu$  and  $\hat{\lambda} \triangleright \hat{\mu}$  if  $\hat{\lambda} \succeq \hat{\mu}$  and  $\hat{\lambda} \neq \hat{\mu}$  for  $\hat{\lambda}, \hat{\mu} \in \mathcal{P}_n^\kappa$ . We define the lexicographic orderings on  $\mathcal{P}_n^\kappa$  similarly.

**Example 5.12.** Suppose  $\hat{\lambda} = (\dots | 0 | 0 | 0 | 4, 3, 1 | 2, 1 | 3, 3)$ . Then  $\lambda = (4, 3, 1 | 2, 1 | 3, 3)$  and

$$\hat{s} = \left( \dots \left| \emptyset \right| \emptyset \left| \begin{array}{|c|c|c|c|} \hline 1 & 8 & 13 & 16 \\ \hline 7 & 12 & 15 & \\ \hline 10 & & & \\ \hline \end{array} \right| \begin{array}{|c|c|} \hline 2 & 6 \\ \hline 3 & \\ \hline \end{array} \left| \begin{array}{|c|c|c|} \hline 4 & 5 & 11 \\ \hline 9 & 14 & 17 \\ \hline \end{array} \right) \in \text{Std}(\hat{\lambda}),$$

and

$$s = \left( \begin{array}{|c|c|c|c|} \hline 1 & 8 & 13 & 16 \\ \hline 7 & 12 & 15 & \\ \hline 10 & & & \\ \hline \end{array} \left| \begin{array}{|c|c|} \hline 2 & 6 \\ \hline 3 & \\ \hline \end{array} \right| \begin{array}{|c|c|c|} \hline 4 & 5 & 11 \\ \hline 9 & 14 & 17 \\ \hline \end{array} \right) \in \text{Std}(\lambda).$$

Suppose  $\Lambda \in P_+$  and  $\lambda = (\lambda^{(\ell)}, \dots, \lambda^{(1)}) \in \mathcal{P}_n^\Lambda$ . Then for any  $s, t \in \text{Std}(\lambda)$ , in Definition 2.33 we have defined  $\hat{\psi}_{st}$  and  $\psi_{st} = \hat{\psi}_{st} + N_n^\Lambda \in \mathcal{R}_n^\Lambda$ . For any standard affine tableau  $\hat{s}, \hat{t}$  we define  $\psi_{\hat{s}\hat{t}} = \hat{\psi}_{st}$ . Also we can define  $\psi_{\hat{s}\hat{t}}^* = \psi_{\hat{t}\hat{s}}$ .

**Example 5.13.** Suppose  $\kappa = (\dots, 0, 2, 1, 0, 2, 1, 0, 2, 1)$  as in Example 5.11. For

$$\hat{s} = \left( \dots \left| \emptyset \right| \emptyset \left| \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \right| \begin{array}{|c|} \hline 6 \\ \hline \end{array} \right) \quad \hat{t} = \left( \dots \left| \emptyset \right| \emptyset \left| \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \right| \begin{array}{|c|} \hline 6 \\ \hline \end{array} \right),$$

with

$$s = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \left| \begin{array}{|c|} \hline 6 \\ \hline \end{array} \right) \quad t = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \left| \begin{array}{|c|} \hline 6 \\ \hline \end{array} \right) \right).$$

Then  $\psi_{\hat{s}\hat{t}} = \hat{\psi}_{st} = e(012211)y_2y_3^2y_5\psi_5\psi_3 \in \mathcal{R}_n$ .

The next Lemma is a straightforward application of the definition of  $\psi_{\hat{s}\hat{t}}$  and  $\deg(\hat{s})$ .

**Lemma 5.14.** Suppose  $\hat{\lambda} \in \mathcal{P}_n^\kappa$  and  $\hat{s}, \hat{t} \in \text{Std}(\hat{\lambda})$ . Then  $\psi_{\hat{s}\hat{t}}$  are homogeneous elements of  $\mathcal{R}_n$  and  $\deg(\psi_{\hat{s}\hat{t}}) = \deg(\hat{s}) + \deg(\hat{t})$ .

Fix  $\alpha \in Q_+$ , a standard sequence  $\Lambda^\infty$  and an inverse multicharge sequence  $\kappa$  corresponding to  $\Lambda^\infty$ . Define a set of homogeneous elements of  $\mathcal{R}_\alpha$  by

$$\mathcal{B}_\alpha^{\Lambda^\infty} = \{ \psi_{\hat{s}\hat{t}} \mid \hat{\lambda} \in \mathcal{P}_n^\kappa, \hat{s}, \hat{t} \in \text{Std}(\hat{\lambda}), \text{res}(\hat{t}) \in I^\alpha \}.$$

Note that by definition  $\mathcal{B}_\alpha^{\Lambda^\infty}$  depends on the choice of  $\kappa$  and hence on  $\Lambda^\infty$ . Remarkably, the main results of this section are true for any inverse multicharge sequence corresponding to  $\Lambda^\infty$ .

**Proposition 5.15.** *The set  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is a homogeneous basis of  $\mathcal{R}_\alpha$ .*

*Proof.* By Lemma 5.14, all of the elements of  $\mathcal{B}_\alpha^{\Lambda^\infty}$  are homogeneous. So we only have to prove that  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is a basis of  $\mathcal{R}_\alpha$ . First of all we show that  $\mathcal{B}_\alpha^{\Lambda^\infty}$  spans  $\mathcal{R}_\alpha$ .

If  $r \in \mathcal{R}_\alpha$ , then we can write  $r$  as a linear combination of homogeneous elements, i.e.  $r = \sum_{i \in \mathbb{N}} c_i r_i$ , where  $c_i \in \mathbb{Z}$ ,  $\deg(r_i) = i$  and there are only finite many  $i \in \mathbb{N}$  with  $c_i \neq 0$ . So it is enough to prove that any homogeneous element  $r \in \mathcal{R}_\alpha$  is a linear combination of  $\mathcal{B}_\alpha^{\Lambda^\infty}$ .

For any  $\Lambda < \Lambda'$ , it is obvious that  $N_\alpha^{\Lambda'} \subseteq N_\alpha^\Lambda$ . Moreover,  $N_\alpha^{\Lambda'}$  is a  $\mathcal{R}_\alpha$ -ideal of  $N_\alpha^\Lambda$ . Hence we can define an infinite filtration

$$\mathcal{R}_\alpha > N_\alpha^{\Lambda^{(1)}} > N_\alpha^{\Lambda^{(2)}} > N_\alpha^{\Lambda^{(3)}} > \dots$$

By Proposition 5.8,  $\lim_{k \rightarrow \infty} m_\alpha^{\Lambda^{(k)}} = \infty$ , so if  $r \in \mathcal{R}_\alpha$  is homogeneous then there exists an integer  $k(r)$  such that  $m_\alpha^{\Lambda^{(k)}} > \deg(r)$  whenever  $k > k(r)$ . Fix  $k > k(r)$  and hence  $r \notin N_\alpha^{\Lambda^{(k)}}$ .

By Proposition 5.5, choosing a multicharge  $\kappa$  corresponding to  $\Lambda$ ,  $\mathcal{R}_\alpha^\Lambda \cong \mathcal{R}_\alpha / N_\alpha^\Lambda$  has a homogeneous basis  $\{ \psi_{\text{st}} \mid \lambda \in \mathcal{P}_n^\Lambda, \text{s}, \text{t} \in \text{Std}(\lambda), \text{res}(\text{t}) \in I^\alpha \}$ . Fix a multicharge  $(\kappa_{\ell_k}, \kappa_{\ell_{k-1}}, \dots, \kappa_2, \kappa_1)$  corresponding  $\Lambda^{(k)}$ . For any homogeneous element  $r \in \mathcal{R}_\alpha$ , there exists  $c_{\text{st}} \in \mathbb{Z}$  with  $\text{res}(\text{t}) \in I^\alpha$  such that

$$\begin{aligned} r + N_\alpha^{\Lambda^{(k)}} &= \sum_{\text{s}, \text{t}} c_{\text{st}} \psi_{\text{st}} = \sum_{\text{s}, \text{t}} c_{\text{st}} \hat{\psi}_{\text{st}} + N_\alpha^{\Lambda^{(k)}} = \sum_{\hat{\text{s}}, \hat{\text{t}}} c_{\text{st}} \psi_{\hat{\text{s}}\hat{\text{t}}} + N_\alpha^{\Lambda^{(k)}} \\ \Rightarrow \quad r - \sum_{\hat{\text{s}}, \hat{\text{t}}} c_{\text{st}} \psi_{\hat{\text{s}}\hat{\text{t}}} &\in N_\alpha^{\Lambda^{(k)}}. \end{aligned}$$

But as  $r$  is a homogeneous element which is not in  $N_\alpha^{\Lambda^{(k)}}$ , we must have  $r - \sum_{\hat{\text{s}}, \hat{\text{t}}} c_{\text{st}} \psi_{\hat{\text{s}}\hat{\text{t}}} = 0$ , i.e.  $r = \sum_{\hat{\text{s}}, \hat{\text{t}}} c_{\text{st}} \psi_{\hat{\text{s}}\hat{\text{t}}}$  with  $\text{res}(\hat{\text{t}}) = \text{res}(\text{t}) \in I^\alpha$ . This shows that  $r$  belongs to the span of  $\mathcal{B}_\alpha^{\Lambda^\infty}$ . Hence  $\mathcal{R}_\alpha$  is spanned by  $\mathcal{B}_\alpha^{\Lambda^\infty}$ .

Next we will prove that  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is linearly independent. Suppose  $S_\alpha$  is a finite subset of  $\mathcal{B}_\alpha^{\Lambda^\infty}$ . Write  $m_{S_\alpha} = \max \{ \deg(\psi_{\hat{\text{s}}\hat{\text{t}}}) \mid \psi_{\hat{\text{s}}\hat{\text{t}}} \in S_\alpha \}$ . By Proposition 5.8 there exists  $k$  such that  $m_\alpha^{\Lambda^{(k)}} > m_{S_\alpha}$ . Hence  $\psi_{\hat{\text{s}}\hat{\text{t}}} \notin N_\alpha^{\Lambda^{(k)}}$  for any  $\psi_{\hat{\text{s}}\hat{\text{t}}} \in S_\alpha$ . This means that for any  $\psi_{\hat{\text{s}}\hat{\text{t}}} \in S_\alpha$ ,  $\psi_{\text{st}} \in \mathcal{B}_\alpha^{\Lambda^{(k)}}$  is nonzero. As by the definition,  $\{ \psi_{\text{st}} \mid \psi_{\hat{\text{s}}\hat{\text{t}}} \in S_\alpha \}$  is a subset of the basis of  $\mathcal{R}_\alpha^{\Lambda^{(k)}}$ . We have  $\sum_{\psi_{\hat{\text{s}}\hat{\text{t}}} \in S_\alpha} c_{\hat{\text{s}}\hat{\text{t}}} \psi_{\hat{\text{s}}\hat{\text{t}}} \in N_\alpha^{\Lambda^{(k)}}$  if and only if  $\sum_{\psi_{\hat{\text{s}}\hat{\text{t}}} \in S_\alpha} c_{\hat{\text{s}}\hat{\text{t}}} \psi_{\text{st}} = 0$  if and only if all  $c_{\hat{\text{s}}\hat{\text{t}}} = 0$ . But  $\psi_{\hat{\text{s}}\hat{\text{t}}} \notin N_\alpha^{\Lambda^{(k)}}$  for any  $\psi_{\hat{\text{s}}\hat{\text{t}}} \in S_\alpha$ , the above result yields that  $\sum_{\psi_{\hat{\text{s}}\hat{\text{t}}} \in S_\alpha} c_{\hat{\text{s}}\hat{\text{t}}} \psi_{\hat{\text{s}}\hat{\text{t}}} = 0$  if and only if  $c_{\hat{\text{s}}\hat{\text{t}}} = 0$ . This shows that  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is linearly independent. Hence  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is a basis of  $\mathcal{R}_\alpha$ .  $\square$

Notice that in the definition of  $\mathcal{B}_\alpha^{\Lambda^\infty}$ , it is well-defined for any inverse multicharge sequence  $\kappa$  corresponds to  $\Lambda^\infty$ . Hence for any weight  $\Lambda$  with  $\ell = l(\Lambda)$ , by the definition of the standard sequence, we can set  $\Lambda^{(1)} = \Lambda$ . Therefore, we obtain a subset of  $\mathcal{B}_\alpha^{\Lambda^\infty}$ :

$$\mathcal{B}_\Lambda^{\Lambda^\infty} = \{ \psi_{\hat{s}\hat{t}} \mid \hat{\lambda} \in \mathcal{P}_n^\kappa \text{ with } l(\hat{\lambda}) \leq \ell, \hat{s}, \hat{t} \in \text{Std}(\hat{\lambda}), \text{res}(\hat{t}) \in I^\alpha \}.$$

**Corollary 5.16.** *Suppose  $\Lambda$  is a weight with level  $\ell$  and  $\Lambda^\infty$  is a standard sequence with  $\Lambda^{(1)} = \Lambda$ . Then*

$$\mathcal{B}_\alpha^{\Lambda^\infty} \setminus \mathcal{B}_\Lambda^{\Lambda^\infty} = \{ \psi_{\hat{s}\hat{t}} \mid \hat{\lambda} \in \mathcal{P}_n^\kappa \text{ with } l(\hat{\lambda}) > \ell = l(\Lambda), \hat{s}, \hat{t} \in \text{Std}(\hat{\lambda}), \text{res}(\hat{t}) \in I^\alpha \}$$

*is a basis of  $N_\alpha^\Lambda$ .*

*Proof.* By Proposition 5.5,  $\mathcal{R}_\alpha^\Lambda$  has a basis  $\{ \psi_{\text{st}} \mid \lambda \in \mathcal{P}_n^\Lambda, \text{s}, \text{t} \in \text{Std}(\lambda), \text{res}(\text{t}) \in I^\alpha \}$ . It is easy to see that when  $\Lambda^{(1)} = \Lambda$ ,

$$\{ \psi_{\text{st}} \mid \lambda \in \mathcal{P}_n^\Lambda, \text{s}, \text{t} \in \text{Std}(\lambda), \text{res}(\text{t}) \in I^\alpha \} = \{ \psi_{\text{st}} = \psi_{\hat{s}\hat{t}} + N_\alpha^\Lambda \mid \psi_{\hat{s}\hat{t}} \in \mathcal{B}_\Lambda^{\Lambda^\infty} \}.$$

So for  $\psi_{\hat{s}\hat{t}} \in \mathcal{B}_\Lambda^{\Lambda^\infty}$ , we must have  $\psi_{\hat{s}\hat{t}} \notin N_\alpha^\Lambda$ .

Now suppose  $\psi_{\hat{s}\hat{t}} \in \mathcal{B}_\alpha^{\Lambda^\infty} \setminus \mathcal{B}_\Lambda^{\Lambda^\infty}$ . Then  $\hat{s}, \hat{t} \in \text{Std}(\hat{\lambda})$  with  $l(\hat{\lambda}) > \ell$ . By the definition it is obvious that  $\psi_{\hat{s}\hat{t}} \in N_\alpha^\Lambda$  when  $\Lambda^{(1)} = \Lambda$ . Then  $N_\alpha^\Lambda$  is spanned by  $\mathcal{B}_\alpha^{\Lambda^\infty} \setminus \mathcal{B}_\Lambda^{\Lambda^\infty}$ .  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is a basis implies the linearly independence of  $\mathcal{B}_\alpha^{\Lambda^\infty} \setminus \mathcal{B}_\Lambda^{\Lambda^\infty}$ . So  $\mathcal{B}_\alpha^{\Lambda^\infty} \setminus \mathcal{B}_\Lambda^{\Lambda^\infty}$  is a basis of  $N_\alpha^\Lambda$ .  $\square$

Recall for any  $\hat{s}, \hat{t} \in \text{Std}(\hat{\lambda})$  with  $\hat{\lambda} \in \mathcal{P}_n^\kappa$ , we define  $\psi_{\hat{s}\hat{t}}^* = \psi_{\hat{t}\hat{s}}$ . By Proposition 5.15,  $*$  can be defined to be a linear bijection from  $\mathcal{R}_\alpha$  to  $\mathcal{R}_\alpha$ . The next Corollary is straightforward by Corollary 5.16.

**Corollary 5.17.** *Suppose  $*$ :  $\mathcal{R}_\alpha \rightarrow \mathcal{R}_\alpha$  is defined as above. Then it can be restricted to a linear bijection  $*$ :  $N_\alpha^\Lambda \rightarrow N_\alpha^\Lambda$ .*

Now we can prove the main result of this section.

**Proposition 5.18.** *Suppose  $\Lambda^\infty$  is a standard sequence and  $\alpha \in Q_+$ . The set  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is a graded cellular basis of  $\mathcal{R}_\alpha$ .*

*Proof.* Recall Definition 2.20 gives the definition of graded cellular basis. Proposition 5.15 shows that  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is a homogeneous basis of  $\mathcal{R}_\alpha$ . To prove the Theorem we need to establish properties 2.20(b) and 2.20(c) of  $\mathcal{B}_\alpha^{\Lambda^\infty}$ .

Suppose  $a$  is an element of  $\mathcal{R}_\alpha$  and  $\psi_{\hat{s}\hat{t}} \in \mathcal{B}_\alpha^{\Lambda^\infty}$  with  $\hat{s}, \hat{t} \in \text{Std}(\hat{\lambda})$ . We can write  $a = \sum_{i \in \mathbb{N}} c_i a_i$  where  $c_i \in \mathbb{Z}$  and  $a_i$  are homogeneous elements in  $\mathcal{R}_\alpha$  with  $\deg(a_i) = i$ . Define  $d_1 = \deg(\psi_{\hat{s}\hat{t}})$  and  $d_2 = \max\{i \mid c_i \neq 0\}$ . By Proposition 5.8 there exists  $k$  such that  $m_\alpha^{\Lambda^{(k)}} > \max\{d_1, d_2, d_1 + d_2\}$ . This means that  $\psi_{\hat{s}\hat{t}}, a$  and  $\psi_{\hat{s}\hat{t}}a$  are not elements of  $N_\alpha^{\Lambda^{(k)}}$ . This means that  $\psi_{\text{st}} = \psi_{\hat{s}\hat{t}} + N_\alpha^{\Lambda^{(k)}}$ ,  $a + N_\alpha^{\Lambda^{(k)}}$  and  $\psi_{\hat{s}\hat{t}}a + N_\alpha^{\Lambda^{(k)}}$  are nonzero elements of  $\mathcal{R}_\alpha^{\Lambda^{(k)}}$ . By Proposition 5.5 and because  $t$  is a bijection,

$$\begin{aligned} \psi_{\text{st}}(a + N_\alpha^{\Lambda^{(k)}}) &= (\psi_{\hat{s}\hat{t}} + N_\alpha^{\Lambda^{(k)}})(a + N_\alpha^{\Lambda^{(k)}}) = \sum_{v \in \text{Std}(\hat{\lambda})} c_{sv} \psi_{sv} + \sum_{\substack{u, v \in \text{Std}(\hat{\mu}) \\ \hat{\mu} \triangleright \hat{\lambda}}} c_{uv} \psi_{uv} \\ \Rightarrow \psi_{\hat{s}\hat{t}}a + N_\alpha^{\Lambda^{(k)}} &= \sum_{\hat{v} \in \text{Std}(\hat{\lambda})} c_{\hat{s}\hat{v}} \psi_{\hat{s}\hat{v}} + \sum_{\substack{\hat{u}, \hat{v} \in \text{Std}(\hat{\mu}) \\ \hat{\mu} \triangleright \hat{\lambda}}} c_{\hat{u}\hat{v}} \psi_{\hat{u}\hat{v}} + N_\alpha^{\Lambda^{(k)}} \\ \Rightarrow \psi_{\hat{s}\hat{t}}a - \left( \sum_{\hat{v} \in \text{Std}(\hat{\lambda})} c_{\hat{s}\hat{v}} \psi_{\hat{s}\hat{v}} + \sum_{\substack{\hat{u}, \hat{v} \in \text{Std}(\hat{\mu}) \\ \hat{\mu} \triangleright \hat{\lambda}}} c_{\hat{u}\hat{v}} \psi_{\hat{u}\hat{v}} \right) &\in N_\alpha^{\Lambda^{(k)}}. \end{aligned}$$

Since the left hand side of the above equation is homogeneous to  $d_1 + d_2$  and  $m_\alpha^{\Lambda^{(k)}} > d_1 + d_2$ , we see that

$$\psi_{\hat{s}\hat{t}}a = \sum_{\hat{v} \in \text{Std}(\hat{\lambda})} c_{\hat{s}\hat{v}} \psi_{\hat{s}\hat{v}} + \sum_{\substack{\hat{u}, \hat{v} \in \text{Std}(\hat{\mu}) \\ \hat{\mu} \triangleright \hat{\lambda}}} c_{\hat{u}\hat{v}} \psi_{\hat{u}\hat{v}}.$$

which shows that  $\mathcal{B}_\alpha^{\Lambda^\infty}$  satisfies 2.20(b).

For 2.20(c), choose arbitrary  $\psi_{\hat{s}\hat{t}}, \psi_{\hat{u}\hat{v}} \in \mathcal{B}_\alpha^{\Lambda^\infty}$ . Suppose  $\deg(\psi_{\hat{s}\hat{t}}) = k_1$  and  $\deg(\psi_{\hat{u}\hat{v}}) = d_2$ . By Proposition 5.8 we can choose  $k$  so that  $m_\alpha^{\Lambda^{(k)}} > \max\{k_1, k_2, k_1 + k_2\}$ . Then by Corollary 5.17,

$$\begin{aligned} (\psi_{\text{st}} \psi_{\text{uv}})^* &= ((\psi_{\hat{s}\hat{t}} + N_\alpha^{\Lambda^{(k)}})(\psi_{\hat{u}\hat{v}} + N_\alpha^{\Lambda^{(k)}}))^* = (\psi_{\hat{s}\hat{t}} \psi_{\hat{u}\hat{v}} + N_\alpha^{\Lambda^{(k)}})^* = (\psi_{\hat{s}\hat{t}} \psi_{\hat{u}\hat{v}})^* + N_\alpha^{\Lambda^{(k)}}, \\ \psi_{vu} \psi_{ts} &= (\psi_{\hat{v}\hat{u}} + N_\alpha^{\Lambda^{(k)}})(\psi_{\hat{t}\hat{s}} + N_\alpha^{\Lambda^{(k)}}) = \psi_{\hat{v}\hat{u}} \psi_{\hat{t}\hat{s}} + N_\alpha^{\Lambda^{(k)}}, \end{aligned}$$

which implies that  $(\psi_{\hat{s}\hat{t}} \psi_{\hat{u}\hat{v}})^* - \psi_{\hat{v}\hat{u}} \psi_{\hat{t}\hat{s}} = N_\alpha^{\Lambda^{(k)}}$ . Then because  $m_\alpha^{\Lambda^{(k)}} > k_1 + k_2$ ,  $(\psi_{\hat{s}\hat{t}} \psi_{\hat{u}\hat{v}})^* - \psi_{\hat{v}\hat{u}} \psi_{\hat{t}\hat{s}} = 0$ , i.e.  $(\psi_{\hat{s}\hat{t}} \psi_{\hat{u}\hat{v}})^* = \psi_{\hat{v}\hat{u}} \psi_{\hat{t}\hat{s}}$ . Because  $*$  is a linear bijection and  $\mathcal{B}_\alpha^{\Lambda^\infty}$  is a basis of  $\mathcal{R}_\alpha$ , this shows that  $*$ :  $\mathcal{R}_\alpha \rightarrow \mathcal{R}_\alpha$  is an anti-isomorphism. Hence  $*$  satisfies 2.20(c). This completes the proof.  $\square$

Combining the above two Propositions and Corollary 5.16 we can get the following results.

**Theorem 5.19.** *For any standard sequence  $\Lambda^\infty$ , the set*

$$\mathcal{B}_n^{\Lambda^\infty} = \{ \psi_{\hat{s}\hat{t}} \mid \hat{\lambda} \in \mathcal{P}_n^K, \hat{s}, \hat{t} \in \text{Std}(\hat{\lambda}) \}$$

*is a graded cellular basis of  $\mathcal{R}_n$ .*

*Proof.* By definition we have  $\mathcal{B}_n^{\Lambda^\infty} = \bigoplus_{\alpha \in Q_+} \mathcal{B}_\alpha^{\Lambda^\infty}$  and  $\mathcal{R}_n = \bigoplus_{\alpha \in Q_+} \mathcal{R}_\alpha$ . By the relations of  $\mathcal{R}_n$  we see that  $\mathcal{R}_\alpha$  are subalgebras. The Theorem follows by Proposition 5.18 straightforward.  $\square$

**Corollary 5.20.** *Suppose  $\Lambda$  is a weight with level  $\ell$  and  $\Lambda^\infty$  is a standard sequence with  $\Lambda^{(1)} = \Lambda$ . Then*

$$\{ \psi_{\hat{s}\hat{t}} \mid \hat{\lambda} \in \mathcal{P}_n^K \text{ with } l(\hat{\lambda}) > \ell = l(\Lambda), \hat{s}, \hat{t} \in \text{Std}(\hat{\lambda}) \}$$

*is a basis of  $N_n^\Lambda$ .*

#### 5.4. Graded simple $\mathcal{R}_n$ -modules

Theorem 5.19 gives a graded cellular basis of  $\mathcal{R}_n$ . Graham and Lehrer [7] described a complete set of irreducible representations of any finite dimensional cellular algebra, however, their results do not apply to  $\mathcal{R}_n$  because it is an infinite dimensional algebra. In this subsection we use Graham and Lehrer's results to construct a complete set of graded simple  $\mathcal{R}_n$ -modules. The graded simple  $\mathcal{R}_n$ -modules have been described by Brundan and Kleshchev [4, Theorem 5.19]. See Remark 5.35 for more details.

First we need to state some properties of the simple  $\mathcal{R}_n$ -modules. We start by showing that the graded dimension of an simple  $\mathcal{R}_n$ -module is bounded below.

**Lemma 5.21.** *Suppose  $r \in \mathcal{R}_n$  is a homogeneous element. Then  $\deg(r) \geq -n(n-1)$ .*

*Proof.* By (2.40) we have the following basis of  $\mathcal{R}_n$ :

$$\{ \hat{e}(\mathbf{i}) \hat{y}_1^{\ell_1} \hat{y}_2^{\ell_2} \dots \hat{y}_n^{\ell_n} \hat{\psi}_w \mid \mathbf{i} \in I^n, w \in \mathfrak{S}_n, \ell_1, \ell_2, \dots, \ell_n \geq 0 \}.$$

If  $\mathbf{i} \in I^n$ ,  $w \in \mathfrak{S}_n$  and  $\ell_1, \ell_2, \dots, \ell_n \geq 0$ , then

$$\begin{aligned} \deg(\hat{e}(\mathbf{i}) \hat{y}_1^{\ell_1} \hat{y}_2^{\ell_2} \dots \hat{y}_n^{\ell_n} \hat{\psi}_w) &\geq \deg(\hat{e}(\mathbf{i}) \hat{y}_1^{\ell_1} \hat{y}_2^{\ell_2} \dots \hat{y}_n^{\ell_n}) + \deg(\hat{e}(\mathbf{i}) \hat{\psi}_w) \\ &= 2(\ell_1 + \ell_2 + \dots + \ell_n) + \deg(\hat{e}(\mathbf{i}) \hat{\psi}_w) \\ &\geq \deg(\hat{e}(\mathbf{i}) \hat{\psi}_w). \end{aligned}$$

As  $w \in \mathfrak{S}_n$ , we have  $l(w) \leq \frac{n(n-1)}{2}$  and by definition,  $\hat{e}(\mathbf{i}) \hat{\psi}_r \geq -2$  for any  $r$  and  $\mathbf{i} \in I^n$ . Therefore

$$\deg(\hat{e}(\mathbf{i}) \hat{\psi}_w) \geq -2 \times \frac{n(n-1)}{2} = -n(n-1).$$

Hence  $\deg(\hat{e}(\mathbf{i}) \hat{y}_1^{\ell_1} \hat{y}_2^{\ell_2} \dots \hat{y}_n^{\ell_n} \hat{\psi}_w) \geq -n(n-1)$ . This completes the proof.  $\square$

Recall that for  $|\alpha| = \sum_{i \in I} a_i$  and  $\hat{e}_\alpha = \sum_{\mathbf{i} \in I^\alpha} \hat{e}(\mathbf{i}) \in \mathcal{R}_n$ .

**Lemma 5.22.** *Suppose  $S$  is a simple  $\mathcal{R}_n$ -module. Then there exists  $\alpha \in Q_+$  with  $|\alpha| = n$  such that for any  $\beta \in Q_+$  with  $|\beta| = n$ ,  $\hat{e}_\beta S = \delta_{\alpha\beta} S$ .*

*Proof.* Suppose  $S$  is a simple  $\mathcal{R}_n$ -module. Because  $1 = \sum_{\mathbf{j} \in I^n} \hat{e}(\mathbf{j}) = \sum_{\substack{\beta \in Q_+ \\ |\beta|=n}} \hat{e}_\beta$ , we can write  $S = \bigoplus_{\substack{\beta \in Q_+ \\ |\beta|=n}} \hat{e}_\beta S$ . Suppose  $\hat{e}_\alpha S \neq 0$  for some  $\alpha \in Q_+$ . Choose any nonzero element  $s \in S$  and  $\beta \in Q_+$  with  $\beta \neq \alpha$ . By Lemma 5.4,  $\hat{e}_\alpha \mathcal{R}_n \hat{e}_\beta = 0$ . So we must have  $\hat{e}_\beta \cdot s = 0$ . Hence  $\hat{e}_\beta S = 0$ . Therefore  $S = \bigoplus_{\substack{\beta \in Q_+ \\ |\beta|=n}} \hat{e}_\beta S = \hat{e}_\alpha S$ . This completes the proof.  $\square$

It is well-known that the irreducible representations of the affine Hecke algebra are finite dimensional as, by Bernstein, the affine Hecke algebra is finite dimensional over its centre. See for example, Proposition 4.1 and Corollary 4.2 of Grojnowski [8], or Proposition 2.12 of Khovanov-Lauda [13]. The next Proposition gives a different approach.

**Proposition 5.23.** *Suppose  $S$  is a graded simple  $\mathcal{R}_n$ -module and  $\alpha \in Q_+$  is such that  $\hat{e}_\beta S = \delta_{\alpha\beta} S$  for  $\beta \in Q_+$ . If  $\Lambda \in P_+$  with  $m_\alpha^\Lambda > n(n-1)$ , then  $S$  is isomorphic to a graded simple  $\mathcal{R}_n^\Lambda$ -module.*

*Proof.* By Lemma 5.22 there exists  $\alpha \in Q_+$  such that  $\hat{e}_\beta S = \delta_{\alpha\beta} S$  for  $\beta \in Q_+$ . Then we choose an arbitrary nonzero homogeneous element  $s \in S$  and suppose  $\deg(s) = d$ . Now for any nonzero homogeneous element  $t \in S$ , because  $S$  is simple, there exists a homogeneous element  $a \in \mathcal{R}_\alpha$  such that  $t = a \cdot s$ . Therefore

$$\deg(t) = \deg(a \cdot s) = \deg(a) + \deg(s) \geq d - n(n-1)$$

where by Lemma 5.21 we have  $\deg(a) \geq -n(n-1)$ . So for any homogeneous nonzero element  $t \in S$ , we have

$$\deg(t) \geq d - n(n-1). \quad (5.24)$$

Similarly, since for any nonzero homogeneous element  $t \in \mathcal{R}_n$  there exists a homogeneous element  $a \in \mathcal{R}_\alpha$  such that  $s = a \cdot t$ , we have

$$\deg(t) \leq d + n(n-1). \quad (5.25)$$

Combining (5.24) and (5.25), we have  $|\deg(s) - \deg(t)| \leq n(n-1)$  for any nonzero homogeneous element  $t \in S$ . Because  $s$  is chosen arbitrarily, we have

$$|\deg(s) - \deg(t)| \leq n(n-1) \quad (5.26)$$

for any nonzero homogeneous elements  $s, t \in S$ .

Suppose  $\Lambda \in P_+$  with  $m_\alpha^\Lambda > n(n-1)$ . For any homogeneous element  $a \in N_\alpha^\Lambda$  and  $t \in S$ , we have  $a \cdot t = 0$  because  $\deg(a \cdot t) - \deg(t) = \deg(a) > n(n-1)$  and (5.26).

For any  $s \in S$ , we can define a map  $f : \mathcal{R}_n \rightarrow S$  by sending  $a$  to  $a \cdot s$ . It is a homomorphism and it is obvious that  $N_\alpha^\Lambda \subseteq \ker f$ . If  $\beta \in Q_+$  and  $\beta \neq \alpha$ , then by Lemma 5.22 we have  $\hat{e}_\beta \cdot s = 0$ . Therefore  $N_\beta^\Lambda \subseteq \ker f$ . Hence  $N_n^\Lambda \subseteq \ker f$ . Therefore we can consider  $S$  as a simple  $\mathcal{R}_n/N_n^\Lambda$ -module, i.e.  $\mathcal{R}_n^\Lambda$ -module. This completes the proof.  $\square$

**Corollary 5.27.** *Suppose  $S$  is a simple graded  $\mathcal{R}_n$ -module. Then  $S$  is finite-dimensional.*

Building on Ariki's [1] work in the ungraded case, Hu and Mathas [9] constructed all graded simple  $\mathcal{R}_n^\Lambda$ -modules in the sense of Graham-Lehrer [7]. They proved that, up to shift, graded simple  $\mathcal{R}_n^\Lambda$ -modules are labeled by the **Kleshchev multipartitions** of  $n$ , which were introduced by Ariki and Mathas [2]. Readers may also refer to Brundan and Kleshchev [4, (3.27)] (where they are called **restricted multipartitions**).

Suppose  $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(l)}) \in \mathcal{P}_n^\Lambda$  and we consider the Young diagram  $[\lambda]$ . Let  $\gamma = (r, c, l)$  be a node in the Young diagram with residue  $i$ , i.e.  $i \equiv r - c + \kappa_l \pmod{e}$ . Then  $\gamma$  is an addable  $i$ -node if  $\gamma \notin [\lambda]$  and  $[\lambda] \cup \{\gamma\}$  is the Young diagram of a multipartition, and  $\gamma$  is a removable  $i$ -node if  $\gamma \in [\lambda]$  and  $[\lambda] \setminus \{\gamma\}$  is the Young diagram of a multipartition.

For each  $\lambda \in \mathcal{P}_n^\Lambda$ , we read all addable and removable  $i$ -nodes in the following order: we start with the first row of  $\lambda^{(1)}$ , and then read rows in  $\lambda^{(1)}$  downward. We then read the first row of  $\lambda^{(2)}$ , and repeat the same procedure, until we finish reading all rows of  $\lambda$ . We write  $A$  for an addable  $i$ -node, and  $R$  for a removable  $i$ -node. Hence we get a sequence of  $A$  and  $R$ . We then delete  $RA$  as many as possible. For example, if we have a sequence  $RARARRAAARRAR$ , the resulting sequence will be  $- - - - - - - AR - - R$ . The node corresponding to the leftmost  $R$  is the **good  $i$ -node**.

The Kleshchev multipartition can then be defined recursively as follows.

**Definition 5.28.** [1, Definition 2.3] *We declare that  $\emptyset$  is Keshchev. Assume that we have already defined the set of Kleshchev multipartitions up to size  $n-1$ . Let  $\lambda$  be a multipartition of  $n$ . We say that  $\lambda$  is a Kleshchev multipartition if there is a good node  $\gamma$  in  $[\lambda]$  such that if  $[\mu] = [\lambda] \setminus \{\gamma\}$  then  $\mu$  is a Kleshchev multipartition.*

Let  $\mathcal{P}_0^\Lambda$  be the set of Kleshchev multipartitions in  $\mathcal{P}_n^\Lambda$ . Let  $S^\lambda$  be the cell module of  $\mathcal{R}_n^\Lambda$  (it is called the **Specht module** in  $\mathcal{R}_n^\Lambda$ ), which was introduced in subsection 1.2, and  $D^\lambda = S^\lambda / \text{rad } S^\lambda$ . Hu-Mathas [9, Corollary 5.11] gave a set of complete non-isomorphic graded simple  $\mathcal{R}_n^\Lambda$ -modules. We note that the graded simple  $\mathcal{R}_n^\Lambda$ -modules were first constructed by Brundan and Kleshchev [4, Theorem 4.11] giving the same classification but without using cellular algebra techniques.

**Theorem 5.29.** *The set  $\{D^\lambda\langle k \rangle \mid \lambda \in \mathcal{P}_0^\Lambda, k \in \mathbb{Z}\}$  is a complete set of pairwise non-isomorphic graded simple  $\mathcal{R}_n^\Lambda$ -modules.*

We consider  $S^\lambda$  and  $D^\lambda\langle k \rangle$  as  $\mathcal{R}_n$ -modules. The actions of  $\hat{e}(\mathbf{i})$ ,  $\hat{y}_r$  and  $\hat{\psi}_s$  on  $S^\lambda$  and  $D^\lambda\langle k \rangle$  are the same as the actions of  $e(\mathbf{i})$ ,  $y_r$  and  $\psi_s$ . Therefore  $D^\lambda\langle k \rangle$  is a simple  $\mathcal{R}_n$ -module. Hence the modules in Theorem 5.29 are a set of simple  $\mathcal{R}_n$ -modules.

The next Lemma is straightforward by the definition of  $D^\lambda$ .

**Lemma 5.30.** *Suppose  $\lambda, \mu \in \mathcal{P}_n^\Lambda$ . Then  $D^\lambda \cong D^\mu$  as  $\mathcal{R}_n^\Lambda$ -modules if and only if  $D^\lambda \cong D^\mu$  as  $\mathcal{R}_n$ -modules.*

Now we can classify all graded simple  $\mathcal{R}_n$ -modules. Following the definitions in Section 1.2, for each  $\hat{\lambda} \in \mathcal{P}_n^\kappa$  we define the cell module  $S^{\hat{\lambda}}$  of  $\mathcal{R}_n$ , which we also call a **Specht module**, associated with a bilinear form  $\langle \cdot, \cdot \rangle$ . Then we define  $\text{rad } S^{\hat{\lambda}}$  and hence the graded simple module  $D^{\hat{\lambda}} = S^{\hat{\lambda}} / \text{rad } S^{\hat{\lambda}}$ .

**Lemma 5.31.** *Suppose  $\hat{\lambda} \in \mathcal{P}_n^\kappa$  and  $\mu = p_k(\hat{\lambda})$  for some  $k \geq l(\hat{\lambda})$ . Then  $S^\mu \cong S^{\hat{\lambda}}$  as  $\mathcal{R}_n$ -modules.*

*Proof.* This is trivial given the definition of Specht modules in  $\mathcal{R}_n$  and  $\mathcal{R}_n^\Lambda$ . □

The next Corollary is straightforward by Lemma 5.30 and Lemma 5.31.

**Corollary 5.32.** *Suppose  $\hat{\lambda} \in \mathcal{P}_n^\kappa$  and  $\mu = p_k(\hat{\lambda})$  for some  $k \geq l(\hat{\lambda})$ . Then  $D^\mu \cong D^{\hat{\lambda}}$  as  $\mathcal{R}_n$ -modules.*

Hence we can prove the following Lemma.

**Lemma 5.33.** *Suppose  $\hat{\lambda}, \hat{\mu} \in \mathcal{P}_n^\kappa$ . Then  $D^{\hat{\lambda}} \cong D^{\hat{\mu}}$  if and only if  $\hat{\lambda} = \hat{\mu}$ .*

*Proof.* The if part is trivial. Now suppose  $D^{\hat{\lambda}} \cong D^{\hat{\mu}}$ . Choose  $k > \max\{l(\hat{\lambda}), l(\hat{\mu})\}$  and set  $\nu = p_k(\hat{\lambda})$  and  $\sigma = p_k(\hat{\mu})$ . Then by Corollary 5.32 we have  $D^\nu \cong D^\sigma$  as  $\mathcal{R}_n$ -modules. Then Theorem 5.29 and Lemma 5.30 implies  $\nu = \sigma$ . Therefore by the definition of  $k$  we have  $\hat{\lambda} = \hat{\mu}$ . This completes the proof. □

Now we extend the Kleshchev multipartitions to affine multipartitions. Define  $\hat{\lambda} \in \mathcal{P}_n^\kappa$  to be an **affine Kleshchev multipartition** if  $\lambda$  is a Kleshchev multipartition. Let  $\mathcal{P}_0^\kappa$  be the set of all affine Kleshchev multipartitions in  $\mathcal{P}_n^\kappa$ . We can now give a complete set of pairwise non-isomorphic graded simple  $\mathcal{R}_n$ -modules.

**Theorem 5.34.** *The set  $\{D^{\hat{\lambda}}\langle k \rangle \mid \hat{\lambda} \in \mathcal{P}_0^\kappa, k \in \mathbb{Z}\}$  is a complete set of pairwise non-isomorphic graded simple  $\mathcal{R}_n$ -modules.*

*Proof.* By the definition of (affine) Kleshchev multipartitions, [9, Corollary 5.11] and Corollary 5.32,  $D^\lambda\langle k \rangle \cong D^\lambda\langle k \rangle \neq 0$  if and only if  $\hat{\lambda} \in \mathcal{P}_0^\kappa$ .

Suppose  $S$  is a graded simple  $\mathcal{R}_n$ -module. By Lemma 5.22 there exists  $\alpha \in Q_+$  such that  $\hat{e}_\beta S = \delta_{\alpha\beta} S$  for  $\beta \in Q_+$ . Then by Proposition 5.8 we can choose  $i$  such that  $m_\alpha^{(i)} > n(n-1)$  and hence by Proposition 5.23,  $S$  is isomorphic to a graded simple  $\mathcal{R}_n^{\Lambda^{(i)}}$ -module. Therefore, by Theorem 5.29 there exist some  $\mu \in \mathcal{P}_n^{\Lambda^{(i)}}$  and  $k \in \mathbb{Z}$  such that  $S \cong D^\mu\langle k \rangle$  as  $\mathcal{R}_n^{\Lambda^{(i)}}$ -modules, and hence as  $\mathcal{R}_n$ -modules. Suppose  $l(\mu) = \ell$ . We can choose  $\hat{\lambda} \in \mathcal{P}_n^\kappa$  such that  $p_\ell(\hat{\lambda}) = \mu$  with  $l(\hat{\lambda}) \leq \ell$ . By Corollary 5.32 we have  $D^\lambda \cong D^\mu$  as  $\mathcal{R}_n$ -modules. Therefore,  $S \cong D^\lambda\langle k \rangle$ . So

$$\{D^\lambda\langle k \rangle \mid \hat{\lambda} \in \mathcal{P}_0^\kappa, k \in \mathbb{Z}\}$$

is a complete set of graded simple  $\mathcal{R}_n$ -modules.

By Lemma 5.33, the set  $\{D^\lambda\langle k \rangle \mid \hat{\lambda} \in \mathcal{P}_0^\kappa, k \in \mathbb{Z}\}$  is a set of pairwise non-isomorphic graded  $\mathcal{R}_n$ -modules. This completes the proof.  $\square$

**Remark 5.35.** Ariki-Mathas [2] showed that the simple  $H_n$ -modules are indexed by aperiodic multisegments. Khovanov and Lauda [13, 12] also give a classification of the graded simple  $\mathcal{R}_n$ -modules for KLR algebras of arbitrary type. Interested readers may also refer to [15], [4], [16], [18], [25], [11] and [21]. As far as we are aware the construction and classification in Theorem 5.34 is new.

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